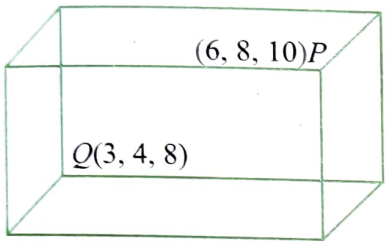
Solutions

Chapter 1

Concept Application Exercises

Exercise 1.1

1.



From the figure it is clear that the length of the edges of the parallelepiped a, b, c are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ or $6 - 3, 8 - 4, 10 - 8$ or $3, 4, 2$.

Therefore, the length of the diagonal will be

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{9 + 16 + 4} = \sqrt{39}.$$

2. We have

$$x^2 - 2x - 3 = 0 \qquad (1)$$

$$\Rightarrow \quad (x - 3)(x + 1) = 0$$

$$\Rightarrow \quad x = 3 \text{ and } x = -1$$

Thus, equation (1) represents combined equation of two planes $x = -1$ and $x = 3$ which are parallel to yz-plane.

Distance between two planes is 4 units.

3. (i) Distance of plane $z = 6$ from xy-plane is 6 units.

Distance of point $P(3, 4, -5)$ from xy-plane is 5 units.

But point P and plane $z = 6$ lie on the opposite sides of xy-plane.

So, distance of point P from plane $z = 6$ is $5 + 6 = 11$ units.

(ii) Distance of plane $x = 7$ from yz-plane is 7 units.

Distance of point $P(3, 4, -5)$ from yz-plane is 3 units.

Since point P and plane $x = 7$ lie on the same side of yz-plane, required distance is $7 - 3 = 4$ units.

4. Point $P(-3, 2, 4)$ lies in second octant.

Image of P in xy-plane is $P'(-3, 2, -4)$ which lies in sixth octant.

Image of P in yz-plane is $P''(3, 2, 4)$ which lies in first octant.

Image of P in xz-plane is $P'''(-3, -2, 4)$ which lies in third octant.

5. Here, $\cos^2 \alpha + \cos^2 (90° - \alpha) + \cos^2 \gamma = 1$

$$\Rightarrow \quad \cos^2 \alpha + \sin^2 \alpha + \cos^2 \gamma = 1$$

$$\Rightarrow \quad \cos^2 \gamma + 1 = 1 \Rightarrow \gamma = 90°$$

6. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$

$$= 2\cos^2 \alpha - 1 + 2\cos^2 \beta - 1 + 2\cos^2 \gamma - 1$$

$$= 2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - 3 = 2 - 3 = -1$$

7. We have point $P(2, 3, 6)$.

$$\therefore \quad \cos \alpha = \frac{2}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2}{7},$$

$$\cos \beta = \frac{3}{7} \text{ and } \cos \gamma = \frac{6}{7}$$

Hence, angle α is greatest as cosine is decreasing function for angles lying in $(0, \pi)$.

Exercise 1.2

1. (i) Let points $(0, 7, -10)$, $(1, 6, -6)$, and $(4, 9, -6)$ be denoted by A, B, and C, respectively.

$$AB = \sqrt{(1 - 0)^2 + (6 - 7)^2 + (-6 + 10)^2}$$

$$= \sqrt{1 + 1 + 16} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(4 - 1)^2 + (9 - 6)^2 + (-6 + 6)^2}$$

$$= \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$CA = \sqrt{(0 - 4)^2 + (7 - 9)^2 + (-10 + 6)^2}$$

$$= \sqrt{(-4)^2 + (-2)^2 + (-4)^2} = \sqrt{36} = 6$$

Here, $AB = BC \neq CA$

Thus, the given points are the vertices of an isosceles triangle.

(ii) Let $(-1, 2, 1)$, $(1, -2, 5)$, $(4, -7, 8)$, and $(2, -3, 4)$ be denoted by A, B, C, and D, respectively.

$$AB = \sqrt{(1+1)^2 + (-2-2)^2 + (5-1)^2}$$

$$= \sqrt{4+16+16} = \sqrt{36} = 6$$

$$BC = \sqrt{(4-1)^2 + (-7+2)^2 + (8-5)^2}$$

$$= \sqrt{9+25+9} = \sqrt{43}$$

$$CD = \sqrt{(2-4)^2 + (-3+7)^2 + (4-8)^2}$$

$$= \sqrt{4+16+16} = \sqrt{36} = 6$$

$$DA = \sqrt{(-1-2)^2 + (2+3)^2 + (1-4)^2}$$

$$= \sqrt{9+25+9} = \sqrt{43}$$

Here, $AB = CD = 6$, $BC = AD = \sqrt{43}$

Hence, the opposite sides of quadrilateral $ABCD$, whose vertices are taken in order, are equal.

Therefore, $ABCD$ is a parallelogram.

Hence, the given points are the vertices of a parallelogram.

2. Let point P be (x, y, z)

Now $x^2 + y^2 + z^2 = 49$ (1)

$y^2 + z^2 = 13$ (2)

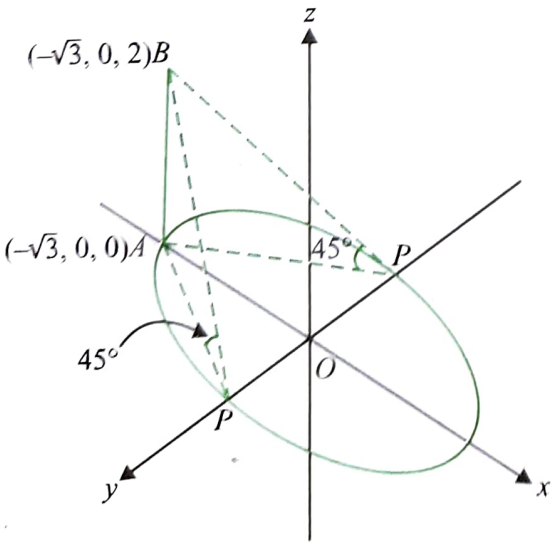
and $x^2 + z^2 = 40$ (3)

$\therefore \quad x^2 = 36, y^2 = 9$ and $z^2 = 4$

$\therefore \quad (x, y, z) \equiv (\pm 6, \pm 3, \pm 2)$

Thus, there are eight points.

3. Clearly, ellipse $x^2 + 3y^2 = 3$ lies in xy-plane.



Point P lies on

$x^2 + 3y^2 = 3$ (1)

Now, from diagram according to given conditions

$AP = AB$

or $(x + \sqrt{3})^2 + (y - 0)^2 = 4$

or $(x + \sqrt{3})^2 + y^2 = 4$ (2)

Solving (1) and (2), we get $x = 0$ and $y = \pm 1$.

So, point P has coordinates $(0, \pm 1, 0)$.

4. We have $P(3, y, z)$. This point lies on the plane $x = 3$, which is parallel to yz-plane.

Given that $OP = 5$, where O is origin.

$\therefore \quad 9 + y^2 + z^2 = 25$

or $y^2 + z^2 = 16$.

This is equation of circle of radius 4 units and centre at $(3, 0, 0)$ lying in the plane $x = 3$.

Exercise 1.3

1. Let C divide AB in the ratio $\lambda : 1$. Then

$$C \equiv \left(\frac{5\lambda+3}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{-6\lambda-4}{\lambda+1}\right) \equiv (9, 8, -10)$$

Comparing x-coordinate, we get

$5\lambda + 3 = 9\lambda + 9$ or $4\lambda = -6$

$\therefore \quad \lambda = -3/2$

Also, from $4\lambda + 2 = 8\lambda + 8$ and $-6\lambda - 4 = -10\lambda - 10$, we get the same value of λ.

Thus, points are collinear.

2. Let A and B be the points that trisect the line segment joining points $P(4, 2, -6)$ and $Q(10, -16, 6)$.

Point A divides PQ in the ratio $1 : 2$. Therefore, by section formula, the coordinates of point A are given by

$$\left(\frac{1(10)+2(4)}{1+2}, \frac{1(-16)+2(2)}{1+2}, \frac{1(6)+2(-6)}{1+2}\right)$$

$$= (6, -4, -2)$$

Point B divides PQ in the ratio $2 : 1$. Therefore, by section formula, the coordinates of point B are given by

$$\left(\frac{2(10)+1(4)}{2+1}, \frac{2(-16)+1(2)}{2+1}, \frac{2(6)-1(6)}{2+1}\right)$$

$$= (8, -10, 2)$$

Thus, (6, –4, –2) and (8, –10, 2) are the points that trisect the line segment joining points P (4, 2, –6) and Q(10, –16, 6).

3. Let given plane divides the join of A and B at point P in the ratio $\lambda : 1$.

$$\therefore \quad P \equiv \left(\frac{7\lambda+2}{\lambda+1}, \frac{2\lambda+3}{\lambda+1}, \frac{-4}{\lambda+1}\right)$$

Point P lies on the given plane.

So, $\dfrac{7\lambda+2}{\lambda+1} - \dfrac{2\lambda+3}{\lambda+1} - \dfrac{8}{\lambda+1} - 7 = 0$

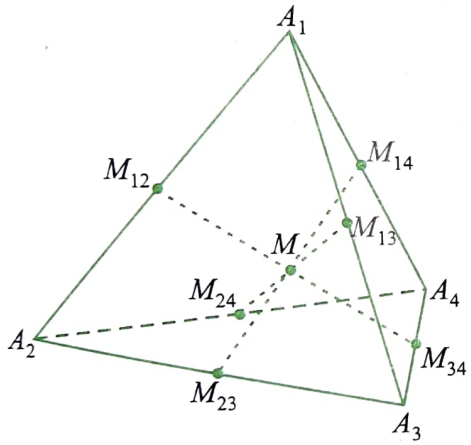
or $5\lambda - 9 - 7\lambda - 7 = 0$

or $\lambda = -8$

Thus point P divides AB externally in the ratio 8 : 1.

So, points A and B lies on the same side of the plane.

4. Consider tetrahedron $A_1A_2A_3A_4$ with $A_i \equiv (x_i, y_i, z_i)$; $i = 1, 2, 3, 4$.



We have three pairs of opposite edges, (A_1A_2, A_3A_4), (A_1A_3, A_2A_4) and (A_1A_4, A_2A_3).

Midpoint of A_1A_2 is

$$M_{12}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

Midpoint of A_3A_4 is

$$M_{34}\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}, \frac{z_3+z_4}{2}\right).$$

Now, midpoint of $M_{12}M_{34}$ is

$$M\left(\frac{\frac{x_1+x_2}{2}+\frac{x_3+x_4}{2}}{2}, \frac{\frac{y_1+y_2}{2}+\frac{y_3+y_4}{2}}{2}, \frac{\frac{z_1+z_2}{2}+\frac{z_3+z_4}{2}}{2}\right)$$

$$\equiv M\left(\frac{\sum_{i=1}^{4} x_i}{4}, \frac{\sum_{i=1}^{4} y_i}{4}, \frac{\sum_{i=1}^{4} z_i}{4}\right)$$

We can show that point M also bisect the line segment joining midpoints of other two pairs of opposite edges.

Exercise 1.4

1. Equation of the sphere whose centre is (3, 1, 2) and radius is 5 is

$$(x-3)^2 + (y-1)^2 + (z-2)^2 = 5^2$$

or $x^2 + y^2 + z^2 - 6x - 2y - 4z - 11 = 0$

2. Let the equation of the sphere passing through (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0,1) be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (1)$$

As (1) passes through (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1) we must have d = 0 ; $1 + 2u + d = 0$; $1 + 2v + d = 0$; $1 + 2w + d = 0$

Since $d = 0$, we get $2u = 2v = 2w = -1$.

Thus, the equation of the required sphere is $x^2 + y^2 + z^2 - x - y - z = 0$.

3. Let $P(x, y, z)$ be any point on the locus. Then

$$PA^2 + PB^2 + PC^2 = 120$$

$\Rightarrow$ $(x-1)^2 + (y-2)^2 + (z-3)^2 + (x-2)^2 + (y+3)^2 + (z-5)^2 + (x-0)^2 + (y-7)^2 + (z-4)^2 = 120$

or $3x^2 + 3y^2 + 3z^2 - 6x - 12y - 24z + 117 = 120$

or $x^2 + y^2 + z^2 - 2x - 4y - 8z - 1 = 0$

This represents a sphere with centre at (1, 2, 4) and radius as $\sqrt{1^2 + 2^2 + 4^2 + 1} = \sqrt{22}$.

4. Since $3PA = 2PB$, we get

$$9PA^2 = 4PB^2$$

$\Rightarrow$ $9[(x-1)^2 + (y-3)^2 + (z-4)^2] = 4[(x-1)^2 + (y+2)^2 + (z+1)^2]$

$\Rightarrow$ $9[x^2 + y^2 + z^2 - 2x - 6y - 8z + 26] = 4[x^2 + y^2 + z^2 - 2x + 4y + 2z + 6]$

$\Rightarrow$ $5x^2 + 5y^2 - 10x - 70y - 80z + 210 = 0$

$\Rightarrow$ $x^2 + y^2 + z^2 - 2x - 14y - 16z + 42 = 0$

This represents a sphere with centre at (1, 7, 8) and radius equal to $\sqrt{1^2 + 7^2 + 8^2 - 42} = \sqrt{72} = 6\sqrt{2}$.

Exercises

Single Correct Answer Type

1. **(3)** Given, point is $P(3, 4, 5)$.

Distance of P from xy-plane = |z coordinate of P| = 5

2. (2) We know that, on the y-axis, $x = 0$ and $z = 0$

Thus, foot of perpendicular from P on y-axis is $A(0, 4, 0)$.

$$\therefore \quad PA = \sqrt{(0-3)^2 + (4-4)^2 + (0-5)^2}$$
$$= \sqrt{9 + 0 + 25} = \sqrt{34}$$

3. (1) We know that, on the xy-plane $z = 0$.

Hence, the coordinates of the points L are $(-1, 7, 0)$.

4. (1) Clearly, $A \equiv (1, 0, 0)$, $B \equiv (0, 2, 0)$ and $C \equiv (0, 0, 3)$.

$\therefore$ Centroid of triangle ABC is

$$\therefore \quad G \equiv \left(\frac{1+0+0}{3}, \frac{0+2+0}{3}, \frac{0+0+3}{3}\right)$$
$$\equiv \left(\frac{1}{3}, \frac{2}{3}, 1\right)$$
$$OG \equiv \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + (1)^2} \equiv \frac{\sqrt{14}}{3}$$

5. (2) Clearly $A \equiv (1, 2, 0)$, $B \equiv (0, 2, 3)$ and $C \equiv (1, 0, 3)$.

$\therefore$ Centroid of triangle ABC is

$$G \equiv \left(\frac{1+0+1}{3}, \frac{2+2+0}{3}, \frac{0+3+3}{3}\right)$$
$$\equiv \left(\frac{2}{3}, \frac{4}{3}, 2\right)$$
$$\therefore \quad OG = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + (2)^2} = \frac{2\sqrt{14}}{3}$$

6. (3) Suppose R divides PQ in the ratio $k : 1$.

$$\therefore \quad R \equiv \left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$$

It is given that the x–coordinate of point R is 6.

$$\therefore \quad \frac{8k+2}{k+1} = 6 \Rightarrow 8k + 2 = 6k + 6$$
$$\Rightarrow \quad 2k = 4 \Rightarrow k = 2$$

Therefore, the coordinates of point R are given by

$$\left(6, \frac{-3}{2+1}, \frac{10 \times 2 + 4}{2+4}\right) \equiv (6, -1, 4)$$

7. (2) We have $x^2 - 3x - 4 = 0$

$$\Rightarrow \quad (x-4)(x+1) = 0$$
$$\Rightarrow \quad x = 4, x = -1$$

Distance between two parallel planes (parallel to yz-plane) is 5 units.

We have $y^2 - 5y + 6 = 0$

$$\Rightarrow \quad (y-2)(y-3) = 0$$
$$\Rightarrow \quad y = 2, y = 3$$

Distance between two parallel planes (parallel to xz-plane) is 1 unit.

We have $z^2 + 6z + 8 = 0$

$$\Rightarrow \quad (z+2)(z+4) = 0$$
$$\Rightarrow \quad z = -2, z = -4$$

Distance between two parallel planes (parallel to xxy-plane) is 2 unit.

So, volume of cuboid is $5 \times 1 \times 2 = 10$ cu. units.

8. (4) Clearly, there will be eight spheres, one in each octant.

9. (2) Let $P(x, y, z)$. Now under given condition, we get

$$\left[\sqrt{x^2+y^2}\right]^2 + \left[\sqrt{y^2+z^2}\right]^2 + \left[\sqrt{z^2+x^2}\right]^2 = 36$$
$$\Rightarrow \quad x^2 + y^2 + z^2 = 18$$

Then distance from origin to the point (x, y, z) is

$$\sqrt{x^2+y^2+z^2} = \sqrt{18} = 3\sqrt{2}$$

10. (2) Let xy-plane divides the line joining the points $A(-1, 3, 4)$ and $B(2, -5, 6)$ in the ratio $\lambda : 1$ at point P then point P has coordinates $\left(\frac{2\lambda-1}{\lambda+1}, \frac{-5\lambda+3}{\lambda+1}, \frac{6\lambda+4}{\lambda+1}\right)$.

Point P lies on xy-plane.

So, $\frac{6\lambda+4}{\lambda+1} = 0$ or $\lambda = -\frac{2}{3}$

Thus, xy plane divides AB externally in the ratio $2 : 3$.

11. (1) Let $(5, 5, \lambda)$ divide the segment joining $(-1, 3, 2)$ and $(-4, 2, -2)$ in the ratio $k : 1$.

$$\therefore \quad \left(\frac{-4k-1}{k+1}, \frac{2k+3}{k+1}, \frac{-2k+2}{k+1}\right) \equiv (5, 5, \lambda)$$

So, $-4k - 1 = 5k + 5$ and $2k + 3 = 5k + 5$

$$\therefore \quad k = -2/3$$

So, comparing z-coordinate with $k = -2/3$, we get

$$4/3 + 2 = \lambda/3$$
$$\therefore \quad \lambda = 10$$

12. (3) Distance from origin $= \sqrt{1+4+9} = \sqrt{14}$

and distance from y-axis $= \sqrt{1+9} = \sqrt{10}$

13. (4) Distance from x-axis $= \sqrt{y^2+z^2} = \sqrt{4+9} = \sqrt{13}$

Distance from y-axis $= \sqrt{1+9} = \sqrt{10}$

Distance from z-axis $= \sqrt{1+4} = \sqrt{5}$

14. (3) $0=\dfrac{a-2+4}{3}\Rightarrow a=-2$

$$0=\frac{1+b+7}{3}\Rightarrow b=-8$$

and $0=\dfrac{3-5+c}{3}\Rightarrow c=2$

15. (2) For xy-plane, $z=0$.

So, $\dfrac{6\lambda+1}{\lambda+1}=0\Rightarrow\lambda=-\dfrac{1}{6}$

$$\therefore\quad x=\frac{-5+18}{5}=\frac{13}{5},\ y=\frac{-1+24}{5}=\frac{23}{5}$$

16. (1) Let point be (x, y, z). Then

$$x^2+y^2+z^2=(x-a)^2+y^2+z^2$$
$$=x^2+(y-b)^2+z^2$$
$$=x^2+y^2+(z-c)^2$$

Therefore $x=\dfrac{a}{2}$, $y=\dfrac{b}{2}$ and $z=\dfrac{c}{2}$

17. (4) Required direction cosines are

$$\frac{3}{\sqrt{3^2+12^2+4^2}},\frac{12}{\sqrt{3^2+12^2+4^2}},\frac{4}{\sqrt{3^2+12^2+4^2}}$$

i.e., $\dfrac{3}{13},\dfrac{12}{13},\dfrac{4}{13}$.

18. (4) Given $\alpha=45°$, $\beta=60°$.

Now, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$$\therefore\quad \cos^2\gamma=1-\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

$$\Rightarrow\quad \gamma=60°\text{ or }120°.$$

19. (2) Here $AB=BC=CD=DA=7$.

Also, $AC=\sqrt{16+25+81}=\sqrt{122}$

and $BD=\sqrt{64+1+9}=\sqrt{74}$

Thus, points are vertices of rhombus.

20. (4) Centre of the sphere is $C(2, 3, -4)$ and radius is $r=\sqrt{4+9+16-(-7)}=6$.

For point $P(1, 2, 0)$, $CP=\sqrt{1+1+16}=\sqrt{18}<6$

So, point $P(1, 2, 0)$ lies inside the sphere.

For other points, we have $CP>6$.

21. (2) For sphere $x^2+y^2+z^2=36$, centre is $C_1(0, 0, 0)$ and radius is $r_1=6$.

For sphere $x^2+y^2+z^2-20x+84=0$, centre is $C_2(10, 0, 0)$ and radius is $r_2=4$.

$C_1C_2=10$ and $r_1+r_2=6+4=10$

Thus, two spheres touch externally at point $(6, 0, 0)$.

22. (2) Given sphere is $x^2+y^2+z^2-2x-4y+6z-4=0$

Since it cots the xy-plane where $z=0$, equation of circular section is $x^2+y^2-2x-4y-4=0$.

The radius of the circle is $\sqrt{1+4-(-4)}=\sqrt{9}=3$.

23. (1) We have plane $6x+3y-2z=18$.

Coordinates of A, B, C are, respectively, $(3, 0, 0)$, $(0, 6, 0)$ and $(0, 0, -9)$.

Thus, centroid of ΔABC is $\left(\dfrac{3}{3},\dfrac{6}{3},\dfrac{-9}{3}\right)$ or $(1, 2, -3)$.

Chapter 2

Concept Application Exercises

Exercise 2.1

1. The unit vector $\hat{a}$ in the direction of vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ is given by $\hat{a} = \dfrac{\vec{a}}{|a|}$.

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$\therefore \quad \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$$

2. Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\therefore \quad |\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

Hence, the direction cosines of $\vec{a}$ are

$$\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right).$$

3. The given points are $A\ (1, 2, -3)$ and $B\ (-1, -2, 1)$. Therefore,

$$\vec{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + \{1-(-3)\}\hat{k}$$
$$= -2\hat{i} - 4\hat{j} + 4\hat{k}$$

$$\therefore \quad |\vec{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4+16+16}$$
$$= \sqrt{36} = 6$$

Hence, the direction cosines of $\vec{AB}$ are

$$\left(-\frac{2}{6}, -\frac{4}{6}, \frac{4}{6}\right) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right).$$

4. Given points are $P(5\hat{i} + 4\hat{j} + a\hat{k})$ and $Q(-\hat{i} + 2\hat{j} - 2\hat{k})$

Now $\vec{PQ} = (-\hat{i} + 2\hat{j} - 2\hat{k}) - (5\hat{i} + 4\hat{j} + a\hat{k})$

$$= (-6\hat{i} - 2\hat{j} - (2+a)\hat{k})$$

Given $|\vec{PQ}| = 7$

$\therefore \quad \sqrt{36 + 4 + (a+2)^2} = 7$

or $\quad (a+2)^2 = 9$

or $\quad a + 2 = \pm 3$

or $\quad a = -5, 1.$

5. Given points are $A(-3, -2, 0)$, $B(3, -3, 1)$ and $C(5, 0, 2)$. Therefore,

$$\vec{AB} = 6\hat{i} - \hat{j} + \hat{k}$$

and $\quad \vec{AC} = 8\hat{i} + 2\hat{j} + 2\hat{k}$

$$|\vec{AC}| = \sqrt{64 + 4 + 4} = \sqrt{72}$$

Unit vector in direction of $\vec{AB}$ is $\dfrac{6\hat{i} - \hat{j} + \hat{k}}{\sqrt{38}}$

$\therefore$ Required vector

$$= \frac{\sqrt{72}}{\sqrt{38}}(6\hat{i} - \hat{j} + \hat{k}) = \frac{6}{\sqrt{19}}(6\hat{i} - \hat{j} + \hat{k})$$

6. We have,

$$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k} \quad \text{and} \quad \vec{b} = \hat{i} - 2\hat{j} + \hat{k}$$

Let $\vec{c}$ be the resultant of $\vec{a}$ and $\vec{b}$. Then

$$\vec{c} = \vec{a} + \vec{b} = (2+1)\hat{i} + (3-2)\hat{j} + (-1+1)\hat{k}$$
$$= 3\hat{i} + \hat{j}$$

$$\therefore \quad |\vec{c}| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$$

$$\therefore \quad \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{(3\hat{i} + \hat{j})}{\sqrt{10}}$$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is

$$\pm 5 \cdot \vec{c} = \pm 5 \cdot \frac{1}{\sqrt{10}}(3\hat{i} + \hat{j})$$
$$= \pm\frac{3\sqrt{10}\,\hat{i}}{2} \pm \frac{\sqrt{10}}{2}\hat{j}.$$

7. The given points are $A\ (1, -2, -8)$, $B(5, 0, -2)$ and $C(11, 3, 7)$. Therefore,

$$\vec{AB} = (5-1)\hat{i} + (0+2)\hat{j} + (-2+8)\hat{k}$$
$$= 4\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\vec{BC} = (11-5)\hat{i} + (3-0)\hat{j} + (7+2)\hat{k}$$
$$= 6\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\vec{AC} = (11-1)\hat{i} + (3+2)\hat{j} + (7+8)\hat{k}$$
$$= 10\hat{i} + 5\hat{j} + 15\hat{k}$$

$$|\vec{AB}| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16+4+36}$$
$$= \sqrt{56} = 2\sqrt{14}$$

$$|\vec{BC}| = \sqrt{6^2 + 3^2 + 9^2} = \sqrt{36+9+81}$$
$$= \sqrt{126} = 3\sqrt{14}$$

$$|\overrightarrow{AC}| = \sqrt{10^2 + 5^2 + 15^2} = \sqrt{100 + 25 + 225}$$
$$= \sqrt{350} = 5\sqrt{14}$$

$\therefore \quad |\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

Thus, the given points A, B, and C are collinear,

Also $2|\overrightarrow{BC}| = 3|\overrightarrow{AB}|$

Hence, point B divides AC in the ratio 2 : 3.

8. Since the diagonals of a rhombus bisect each other, $\overrightarrow{OA} = -\overrightarrow{OC}$ and $\overrightarrow{OB} = -\overrightarrow{OD}$ and so $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{0}$.

9. Let the position vectors of A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively. Then the position vectors of D, E and F are $(\vec{b} + \vec{c})/2, (\vec{c} + \vec{a})/2$ and $(\vec{a} + \vec{b})/2$, respectively. Therefore,

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \left(\frac{\vec{b} + \vec{c}}{2} - \vec{a}\right) + \left(\frac{\vec{c} + \vec{a}}{2} - \vec{b}\right)$$
$$+ \left(\frac{\vec{a} + \vec{b}}{2} - \vec{c}\right) = \vec{0}$$

10. Since the diagonals of a parallelogram bisect each other, P is the middle point of AC and BD both. Therefore,

$$\overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OP} \text{ and } \overrightarrow{OB} + \overrightarrow{OD} = 2\overrightarrow{OP}$$

11. F is the middle point of BD. Therefore

$$\overrightarrow{AB} + \overrightarrow{AD} = 2\overrightarrow{AF} \qquad \text{(i)}$$

Similarly, $\overrightarrow{CB} + \overrightarrow{CD} = 2\overrightarrow{CE}$ (ii)

Adding (i) and (ii), we get

$$\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD}$$
$$= 2(\overrightarrow{AF} + \overrightarrow{CF}) = -2(\overrightarrow{FA} + \overrightarrow{FC})$$
$$= -2(\overrightarrow{FE}) \text{ (because } E \text{ is the midpoint of } AC)$$
$$= 4\overrightarrow{EF}$$

12. (2) We have, $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$

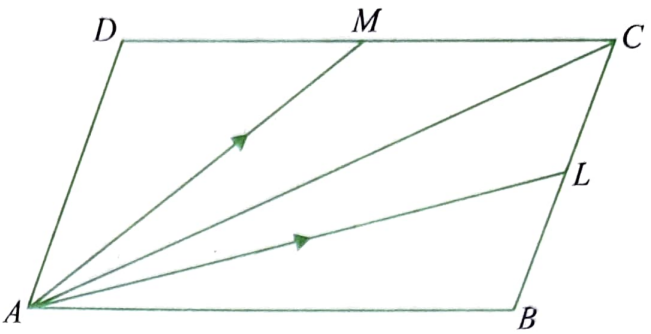
$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{BC}$

Since the initial point of $\overrightarrow{BC}$ is the terminal point of $\overrightarrow{AB}$, A, B and C are collinear.

13. A vector along the internal bisector is

$$\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3} + \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$
$$= \frac{1}{3}(3\hat{i} - \hat{j} + 4\hat{k})$$

14.

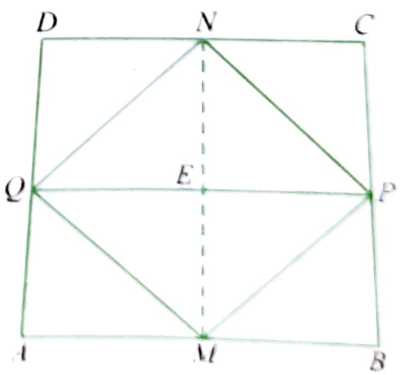


$$\overrightarrow{AL} = \overrightarrow{AB} + \overrightarrow{BL} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$$
$$= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$$
$$\overrightarrow{AM} = \overrightarrow{AD} + \overrightarrow{DM} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC}$$
$$= \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB}$$

Adding, $\overrightarrow{AL} + \overrightarrow{AM} + \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{AD})$

$$= \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{3}{2}\overrightarrow{AC}$$

15. We know that the figure formed by the lines joining the midpoints of the sides of a quadrilateral is a parallelogram. Hence, $MPNQ$ is a parallelogram, whose diagonals are MN and PQ intersecting at E, which is the midpoint of both MN and PQ. For any origin O, we have $\overrightarrow{OA} + \overrightarrow{OB} = 2(\overrightarrow{OM})$ (as M is the midpoint of AB).



$$\overrightarrow{OC} + \overrightarrow{OB} = 2(\overrightarrow{ON})$$

(as N is the midpoint of BC)

$$\Rightarrow \quad \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2(\overrightarrow{OM} + \overrightarrow{ON})$$
$$= 2(2\overrightarrow{OE}) = 4\overrightarrow{OE}$$

where E is the midpoint of MN as it is the intersection of the diagonals of a parallelogram.

16. We have $\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$. Therefore,

$$|\vec{a}| = \sqrt{9+16+4} = \sqrt{29}$$

Therefore, the unit vector parallel to

$$a = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{(29)}}(3\hat{i} + 4\hat{j} - 2\hat{k}).$$

Now suppose $\vec{b}$ is the vector which when added to $\vec{a}$ gives the resultant $\hat{i}$. Then

$$\vec{a} + \vec{b} = \hat{i} \text{ or } \vec{b} = \hat{i} - \vec{a} = \hat{i} - (3\hat{i} + 4\hat{j} - 2\hat{k})$$

$$\therefore \quad \vec{b} = -2\hat{i} - 4\hat{j} + 2\hat{k}$$

17. $|\overrightarrow{OA}| = |\overrightarrow{OB}| = \sqrt{14}$

ΔAOB is isosceles. Hence, the bisector of angle AOB will bisect the base AB.

Hence, P is the midpoint $(2, 2, -2)$ of AB. Therefore,

$$\overrightarrow{OP} = 2(\hat{i} + \hat{j} - \hat{k})$$

18. $\vec{r_3} = p\vec{r_1} + q\vec{r_2}$

$$= \frac{p\vec{r_1} + (1-p)\vec{r_2}}{p + (1-p)}$$

r_3 divides $\vec{r_1}$ and $\vec{r_2}$ in the ratio $(1-p):p$

Hence, r_1, r_2 and r_3 are collinear.

19. $|\vec{a}| = |\vec{b}| = 1$

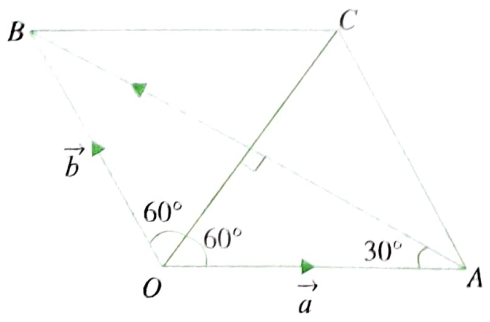
Hence, vectors $\vec{a}$ and $\vec{b}$ are adjacent sides of rhombus $OACB$

Angle between $\vec{a}$ and $\vec{b}$ is 120°.

Also diagonals of rhombus are OC and AB. Thus,

$$\overrightarrow{OC} = \vec{a} + \vec{b} \text{ and } \overrightarrow{AB} = \vec{b} - \vec{a}$$

Clearly from the figure, angle between $\vec{b}$ and $\vec{b} - \vec{a}$ is 30°.



20. A vector bisecting the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|} \pm \frac{\vec{b}}{|\vec{b}|}$.

Here $\frac{2\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}} \pm \frac{\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{6}}$

i.e., $\frac{3\hat{i} - \hat{j}}{\sqrt{6}}$ or $\frac{\hat{i} + 3\hat{j} - 2\hat{k}}{\sqrt{6}}$

A vector of magnitude 3 along these vectors is

$$\frac{3(3\hat{i} - \hat{j})}{\sqrt{10}} \text{ or } \frac{3(\hat{i} + 3\hat{j} - 2\hat{k})}{\sqrt{14}}$$

Exercise 2.2

1. Since $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$

$$3\vec{a} + \vec{c} = 2\vec{b} + 2\vec{d}$$

$$\Rightarrow \quad \frac{3\vec{a} + \vec{c}}{4} = \frac{2\vec{b} + 2\vec{d}}{4} \text{ or } \frac{3\vec{a} + \vec{c}}{3+1} = \frac{\vec{b} + \vec{d}}{2}$$

Therefore, P.V. of the point dividing AC in the ratio 1 : 3 is the same as the P.V. of midpoint of BD.

So AC and BD intersect at P, whose P.V. is $\frac{3\vec{a} + \vec{c}}{4}$ or $\frac{\vec{b} + \vec{d}}{2}$. Point P divides AC in the ratio 3 : 1 and BD in the ratio 1 : 1.

2. Consider $2\vec{a} - \vec{b} + 3\vec{c} = x(\vec{a} + \vec{b} - 2\vec{c})$

$$+ y(\vec{a} + \vec{b} - 3\vec{c})$$

or $2\vec{a} - \vec{b} + 3\vec{c} = (x+y)\vec{a} + (x+y)\vec{b}$

$$+ (-2x - 3y)\vec{c}$$

$x + y = 2$ (i)

$x + y = -1$ (ii)

$-2x - 3y = 3$ (iii)

Multiplying (i) by 3 and adding it to (iii), we get

$x = 9$

From (i), $9 + y = 2$ or $y = -7$

Now putting $x = 9$ and $y = -7$ in (ii), we get

$9 - 7 = -1$

or $2 = -1$, which is not true.

Therefore, the given vectors are not coplanar.

Alternate method:

We have determinant of co-efficients as

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & -3 \end{vmatrix} = -3 \neq 0$$

Hence, vectors are non-coplanar.

3. (i) Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}, \vec{b} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{c} = -\vec{i} - 2\vec{j} + 2\vec{k}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & -2 & 2 \end{vmatrix} = -1$$

Hence, vectors are non-coplanar and linearly independent.

(ii) Let $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}, \vec{b} = 2\vec{i} - \vec{j} + 7\vec{k}$, $\vec{c} = 7\vec{i} - \vec{j} + 13\vec{k}$

$$\begin{vmatrix} 3 & 1 & -1 \\ 2 & -1 & 7 \\ 7 & -1 & 13 \end{vmatrix} = 0$$

Hence, vectors are coplanar and linearly dependent.

4. Putting the values of $\vec{A}$ and $\vec{B}$, and then equating the coefficients of $\vec{a}$ and $\vec{b}$ on both sides, we get

$3(p + 4q) = 2(-2p + q + 2)$

$3(2p + q + 1) = 2(2p - 3q - 1)$

$7p + 10q = 4$ and $2p + 9q = -5$

Solving, we get $p = 2$ and $q = -1$

5. Points $A(\ell_1\vec{a} + m_1\vec{b} + n_1\vec{c}), B(\ell_2\vec{a} + m_2\vec{b} + n_2\vec{c})$, $C(\ell_3\vec{a} + m_3\vec{b} + n_3\vec{c}), D(\ell_4\vec{a} + m_4\vec{b} + n_4\vec{c})$ are coplanar.

$\Rightarrow$ Vectors

$$\vec{AB} = (\ell_1 - \ell_2)\vec{a} + (m_1 - m_2)\vec{b} + (n_1 - n_2)\vec{c},$$

$$\vec{AC} = (\ell_1 - \ell_3)\vec{a} + (m_1 - m_3)\vec{b} + (n_1 - n_3)\vec{c},$$

$$\vec{AD} = (\ell_1 - \ell_4)\vec{a} + (m_1 - m_4)\vec{b} + (n_1 - n_4)\vec{c}$$

are coplanar

$$\Rightarrow \begin{vmatrix} \ell_1 - \ell_2 & m_1 - m_2 & n_1 - n_2 \\ \ell_1 - \ell_3 & m_1 - m_3 & n_1 - n_3 \\ \ell_1 - \ell_3 & m_1 - m_3 & n_1 - n_4 \end{vmatrix} = 0$$

Now if $\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$

Then applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$, we have

$$\begin{vmatrix} \ell_1 & \ell_2 - \ell_1 & \ell_3 - \ell_1 & \ell_4 - \ell_1 \\ m_1 & m_2 - m_1 & m_3 - m_1 & m_4 - m_1 \\ n_1 & n_2 - n_1 & n_3 - n_1 & n_4 - n_1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0$$

or $$\begin{vmatrix} \ell_1 - \ell_2 & m_1 - m_2 & n_1 - n_2 \\ \ell_1 - \ell_3 & m_1 - m_3 & n_1 - n_3 \\ \ell_1 - \ell_3 & m_1 - m_3 & n_1 - n_4 \end{vmatrix} = 0$$

6. Any vector $\vec{r}$ can be uniquely expressed as a linear combination of three non-coplanar vectors.

Let us choose that

$$7\vec{a} - 11\vec{b} + 15\vec{c} = x(\vec{a} - 2\vec{b} + 3\vec{c}) + y(2\vec{a} - 3\vec{b} + 4\vec{c}) + z(3\vec{a} - 4\vec{b} + 5\vec{c})$$

Comparing the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ on both sides, we get

$x + 2y + 3z = 7, -2x - 3y - 4z = -11,$

$3x + 4y + 5z = 15$

Eliminating x and then solving for y and z, we get $x = 1, y = 3, z = 0$

7. Vectors $a\hat{i} + a\hat{j} + c\hat{k}, \hat{i} + \hat{k}, c\hat{i} + c\hat{j} + b\hat{k}$ are coplanar. Thus,

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

or $c^2 - ab = 0$

For equation $ax^2 + 2cx + b = 0$

$D = 4c^2 - 4ab = 4(c^2 - ab) = 0$

Hence, roots are real and equal

Exercises

Single Correct Answer Type

1. **(1)** Four or more than four non-zero vectors are always linearly dependent.

2. **(4)** $3\vec{a} + 4\vec{b} + 5\vec{c} = 0$

Hence, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

No other conclusion can be derived from it.

3. **(3)** $\vec{BC} = \vec{OC} - \vec{OB} = 4\hat{i} + 2\hat{j} - 4\hat{k}$

$\vec{AB} = -3\hat{i} - 3\hat{k}, \vec{AC} = \hat{i} + 2\hat{j} - 7\hat{k}$

$BC^2 = 36, AB^2 = 18, AC^2 = 54$

Clearly, $AC^2 = BC^2 + AB^2$

$\therefore \quad \angle B = 90°$

4. (3) $|\vec{a} + \vec{b}| < |\vec{a} - \vec{b}|$

$\Rightarrow \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}$

5. (3) The position vector of the point O with respect to itself is

$$\frac{\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

$$\Rightarrow \frac{\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C}{\sin 2A + \sin 2B + \sin 2C} = \vec{0}$$

or $\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C = \vec{0}$

6. (1) We have $\overrightarrow{GB} + \overrightarrow{GC} = (1+1)\overrightarrow{GD} = 2\overrightarrow{GD}$, where D is the midpoint of BC. Therefore,

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{GA} + 2\overrightarrow{GD}$$

$$= \overrightarrow{GA} - \overrightarrow{GA} = 0$$

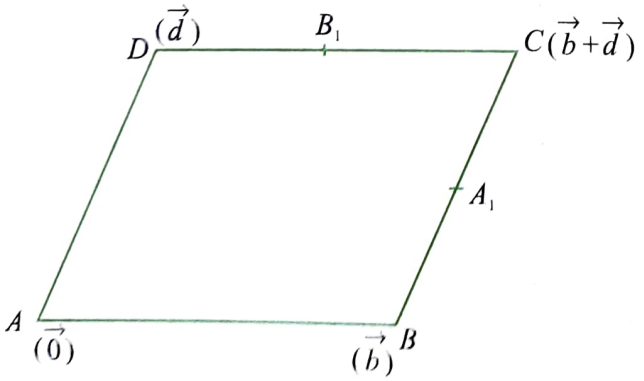
($\because$ G divides AC in the ratio 2 : 1,

$\therefore 2\overrightarrow{GD} = -\overrightarrow{GA}$)

7. (3) $m\vec{a}$ is a unit vector if and only if

$$|m\vec{a}| = 1 \Rightarrow |m|\,|\vec{a}| = 1 \Rightarrow |m|\,a = 1 \Rightarrow a = \frac{1}{|m|}$$

8. (3)



Let P.V. of A, B and D be $\vec{0}, \vec{b}$ and $\vec{d}$, respectively.

Then P.V. of C, $\vec{c} = \vec{b} + \vec{d}$

Also P.V. of $A_1 = \vec{b} + \frac{\vec{d}}{2}$

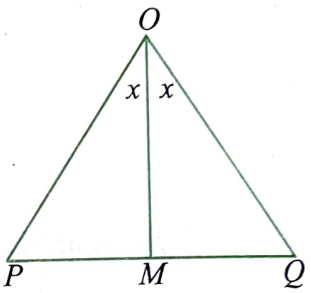
and P.V. of $B_1 = \vec{d} + \frac{\vec{b}}{2}$

$\Rightarrow \quad \overrightarrow{AA_1} + \overrightarrow{AB_1} = \frac{3}{2}(\vec{b} + \vec{d}) = \frac{3}{2}\overrightarrow{AC}$

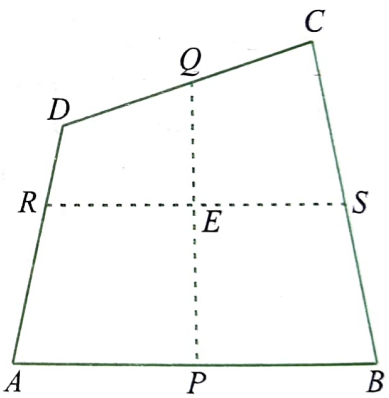
9. (2) Since $|\overrightarrow{OP}| = |\overrightarrow{OQ}| = \sqrt{14}$, ΔOPQ is isosceles.

Hence, the internal bisector OM is perpendicular to PQ and M is the midpoint of P and Q. Therefore,

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ}) = 2\hat{i} + \hat{j} - 2\hat{k}$$



10. (4)



Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OD} = \vec{d}$. Therefore,

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$$

P, the midpoint of AB, is $\frac{\vec{a} + \vec{b}}{2}$

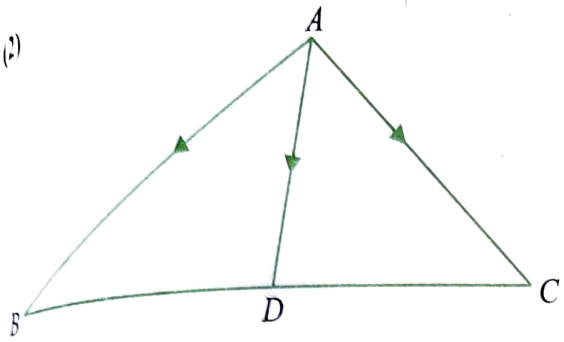
Q, the midpoint of $\overrightarrow{CD}$, is $\frac{\vec{c} + \vec{d}}{2}$

Therefore, the midpoint of $\overrightarrow{PQ}$ is $\frac{\vec{a} + \vec{b} + \vec{c} + \vec{d}}{4}$.

Similarly, the midpoint of RS is $\frac{\vec{a} + \vec{b} + \vec{c} + \vec{d}}{4}$,

i.e., $\overrightarrow{OE} = \frac{\vec{a} + \vec{b} + \vec{c} + \vec{d}}{4} \Rightarrow x = 4.$

11. (2)



$$\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$$

$$\therefore \quad \overrightarrow{AD} = \frac{1}{2}\{(-3\hat{i}+4\hat{k})+(5\hat{i}-2\hat{j}+4\hat{k})\}$$

$$= \hat{i} - \hat{j} + 4\hat{k}$$

Length of $AD = \sqrt{1+1+16} = \sqrt{18}$

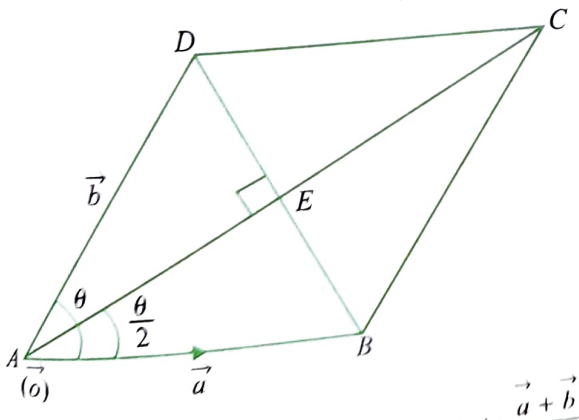
12. (4) $\vec{a} - \vec{b} = 2(\vec{d} - \vec{c})$

$$\therefore \quad \frac{\vec{a}+2\vec{c}}{2+1} = \frac{\vec{b}+2\vec{d}}{2+1}$$

Hence, AC and BD trisect each other as L.H.S. is the position vector of a point trisecting A an C, and R.H.S. that of B and D.

13. (2) Vector in the direction of angular bisector of $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}+\vec{b}}{2}$.

Unit vector in this direction is $\dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$.



From the figure, position vector of E is $\dfrac{\vec{a}+\vec{b}}{2}$

Now in triangle AEB, $AE = AB\cos\dfrac{\theta}{2}$

$$\Rightarrow \left|\frac{\vec{a}+\vec{b}}{2}\right| = \cos\frac{\theta}{2}$$

Hence, unit vector along the bisector is $\dfrac{\vec{a}+\vec{b}}{2\cos(\theta/2)}$

14. (2) $|a| + |b| + |c| = \sqrt{a^2+b^2+c^2}$

$\Leftrightarrow 2|ab| + 2|bc| + 2|ca| = 0$

$\Leftrightarrow ab = bc = ca = 0 \Leftrightarrow$ any two of a, b and c are zero

15. (3) $\vec{\alpha} = \vec{a} + \vec{b} + \vec{c} = 6\hat{i} + 12\hat{j}$

Let $\vec{\alpha} = x\vec{a} + y\vec{b} \Rightarrow 6x + 2y = 6$

and $-3x - 6y = 12$

$\therefore \quad x = 2, y = -3$

$\therefore \quad \vec{\alpha} = 2\vec{a} - 3\vec{b}$

16. (3) Given $\vec{\alpha} + \vec{\beta} + \vec{\gamma} = a\vec{\delta}$ (i)

$\vec{\beta} + \vec{\gamma} + \vec{\delta} = b\vec{\alpha}$ (ii)

From (i), $\vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = (a+1)\vec{\delta}$ (iii)

From (ii), $\vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = (b+1)\vec{\alpha}$ (iv)

From (iii) and (iv), we get

$(a+1)\vec{\delta} = (b+1)\vec{\alpha}$ (v)

Since $\vec{\alpha}$ is not parallel to $\vec{\delta}$,

From (v), $a + 1 = 0$ and $b + 1 = 0$

From (iii), $\vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = 0$

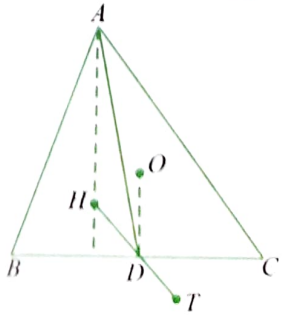
17. (1) Let the origin of reference be the circumcentre of the triangle.

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OT} = \vec{t}$

Then $|\vec{a}| = |\vec{b}| = |\vec{c}| = R$ (circumradius)

Again $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + 2\overrightarrow{OD}$

$= \overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OH}$



Therefore, the P.V. of H is $\vec{a}+\vec{b}+\vec{c}$. Since D is the midpoint of HT, we have

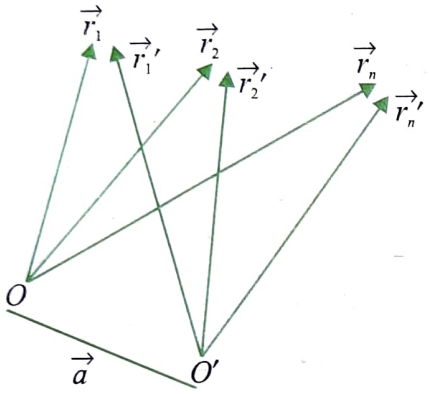
$$\frac{\vec{a}+\vec{b}+\vec{c}+\vec{t}}{2}=\frac{\vec{b}+\vec{c}}{2}\Rightarrow \vec{t}=-\vec{a}$$

$\therefore\ \overrightarrow{AT}=-2\vec{a}$ or $\overrightarrow{AT}=|-2\vec{a}|=2|\vec{a}|=2R.$

But $BC=2R\sin A=R$; therefore,

$$AT=2BC$$

18. (3) Given $a_1\vec{r}_1+a_2\vec{r}_2+\cdots+a_n\vec{r}_n=0$



Now $\vec{a}+\vec{r}_1'=\vec{r}_1$ and so on

Hence,

$$a_1(\vec{a}+\vec{r}_1')+a_2(\vec{a}+\vec{r}_2')+\cdots+a_n(\vec{a}+\vec{r}_n')=0$$

$$a_1\vec{r}_1'+a_2\vec{r}_2'+\cdots+a_n\vec{r}_n'+\vec{a}(a_1+a_2+\cdots+a_n)=0$$

Hence, $a_1\vec{r}_1'+a_2\vec{r}_2'+\cdots+a_n\vec{r}_n'=0$

if $\quad a_1+a_2+\cdots+a_n=0.$

19. (4) $\vec{r}_1+2\vec{r}_2=(p\vec{a}+q\vec{b}+\vec{c})+2(\vec{a}+p\vec{b}+q\vec{c})$

$$=(p+2)\vec{a}+(q+2p)\vec{b}+(1+2q)\vec{c}$$

$$2\vec{r}_1+\vec{r}_2=(2p+1)\vec{a}+(2q+p)\vec{b}+(2+q)\vec{c}$$

$$\frac{p+2}{2p+1}=\frac{q+2p}{2q+p}=\frac{1+2q}{2+q}$$

$$=\frac{p+q+2p+2q+3}{p+q+2p+2q+3}=1$$

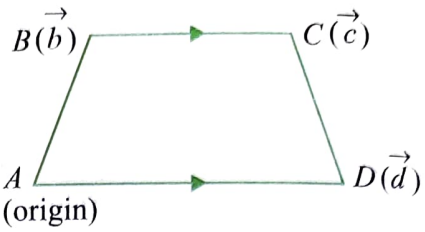
$\Rightarrow \quad p=1$ and $q=1$

20. (4) $\sqrt{3}\tan\theta+1=0$ and $\sqrt{3}\sec\theta-2=0$

$$\Rightarrow\quad \theta=\frac{11\pi}{6}$$

$$\Rightarrow\quad \theta=2n\pi+\frac{11\pi}{6},\ n\in Z$$

21. (4) $\vec{c}-\vec{b}=\alpha\vec{d}$ and $\vec{p}=\overrightarrow{AC}+\overrightarrow{BD}=\mu\overrightarrow{AD}$



Hence, $\vec{p}=\vec{c}+\vec{d}-\vec{b}=\mu\vec{d}$

(using $\vec{c}-\vec{b}=\alpha\vec{d}$)

or $\quad \alpha+1=\mu$

22. (2) Note that $\vec{a}+\vec{b}=\vec{c}$

23. (1) $\hat{a}=\dfrac{-4\hat{i}+3\hat{k}}{5}$; $\hat{b}=\dfrac{14\hat{i}+2\hat{j}-5\hat{k}}{15}$

A vector $\vec{V}$ bisecting the angle between $\vec{a}$ and $\vec{b}$ is

$$\vec{V}=\hat{a}+\hat{b}$$

$$=\frac{-12\hat{i}+9\hat{k}+14\hat{i}+2\hat{j}-5\hat{k}}{15}$$

$$=\frac{2\hat{i}+2\hat{j}+4\hat{k}}{15}$$

Required vector $\vec{d}=\sqrt{6}\,\hat{V}=\hat{i}+\hat{j}+2\hat{k}$

24. (4) We must have $\lambda(\hat{i}-3\hat{j}+5\hat{k})=\hat{a}+\dfrac{2\hat{k}+2\hat{j}-\hat{i}}{3}$.

Therefore,

$$3\hat{a}=3\lambda(\hat{i}-3\hat{j}+5\hat{k})-(2\hat{k}+2\hat{j}-\hat{i})$$

$$=\hat{i}(3\lambda+1)-\hat{j}(2+9\lambda)+\hat{k}(15\lambda-2)$$

or $\quad 3|\hat{a}|=\sqrt{(3\lambda+1)^2+(2+9\lambda)^2+(15\lambda-2)^2}$

or $\quad 9=(3\lambda+1)^2+(2+9\lambda)^2+(15\lambda-2)^2$

or $\quad 315\lambda^2-18\lambda=0\Rightarrow\lambda=0,\ \dfrac{2}{35}$

If $\lambda=0$, $\vec{a}=\hat{i}-2\hat{j}-2\hat{k}$ (not acceptable)

For $\lambda=\dfrac{2}{35}$, $\vec{a}=\dfrac{41}{105}\hat{i}-\dfrac{88}{105}\hat{j}-\dfrac{40}{105}\hat{k}$

25. (3) Suppose the bisector of angle A meets BC at D. Then AD divides BC in the ratio $AB:AC$. So, P.V. of D is given by

$$\frac{|\overrightarrow{AB}|(2\hat{i}+5\hat{j}+7\hat{k})+|\overrightarrow{AC}|(2\hat{i}+3\hat{j}+4\hat{k})}{|\overrightarrow{AB}|+|\overrightarrow{AC}|}$$

But $\overrightarrow{AB} = -2\hat{i}-4\hat{j}-4\hat{k}$

and $\overrightarrow{AC} = -2\hat{i}-2\hat{j}-\hat{k}$

$\Rightarrow |\overrightarrow{AB}| = 6$ and $|\overrightarrow{AC}| = 3$

Therefore, P.V. of D is given by

$$\frac{6(2\hat{i}+5\hat{j}+7\hat{k})+3(2\hat{i}+3\hat{j}+4\hat{k})}{6+3}$$

$$= \frac{1}{3}(6\hat{i}+13\hat{j}+18\hat{k})$$

(2) The point that divides $5\hat{i}$ and $5\hat{j}$ in the ratio of $k:1$ is $\frac{(5\hat{j})k+(5\hat{i})1}{k+1}$

$$\therefore \quad \vec{b} = \frac{5\hat{i}+5k\hat{j}}{k+1}$$

Also, $|\vec{b}| \le \sqrt{37}$

$$\Rightarrow \frac{1}{k+1}\sqrt{25+25k^2} \le \sqrt{37}$$

or $5\sqrt{1+k^2} \le \sqrt{37}(k+1)$

Squaring both sides, we get

$25(1+k^2) \le 37(k^2+2k+1)$

or $6k^2+37k+6 \ge 0$ or $(6k+1)(k+6) \ge 0$

$$k \in (-\infty, -6] \cup \left[-\frac{1}{6}, \infty\right)$$

27. (2) Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$,

then $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{OP} = \frac{1}{3}\vec{a}$,

$\overrightarrow{OQ} = \frac{1}{2}\vec{b}$, $\overrightarrow{OR} = \frac{1}{3}\vec{c}$.

Since P, Q, R and S are coplanar, then

$\overrightarrow{PS} = \alpha\overrightarrow{PQ} + \beta\overrightarrow{PR}$

($\overrightarrow{PS}$ can be written as a linear combination of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$)

$$= \alpha(\overrightarrow{OQ} - \overrightarrow{OP}) + \beta(\overrightarrow{OR} - \overrightarrow{OP})$$

i.e., $\overrightarrow{OS} - \overrightarrow{OP} = -(\alpha+\beta)\frac{\vec{a}}{3} + \frac{\alpha}{2}\vec{b} + \frac{\beta}{3}\vec{c}$

$$\Rightarrow \quad \overrightarrow{OS} = (1-\alpha-\beta)\frac{\vec{a}}{3} + \frac{\alpha}{2}\vec{b} + \frac{\beta}{3}\vec{c} \qquad \text{(i)}$$

Given $\overrightarrow{OS} = \lambda\overrightarrow{AB} = \lambda(\vec{b}-\vec{a})$ (ii)

From (i) and (ii), $\beta = 0$, $\frac{1-\alpha}{3} = -\lambda$ and $\frac{\alpha}{2} = \lambda$

$\Rightarrow \quad 2\lambda = 1 + 3\lambda$

or $\lambda = -1$

28. (1) Let the incentre be at the origin and be $A(\vec{p})$, $B(\vec{q})$ and $C(\vec{r})$. Then

$$\overrightarrow{IA} = \vec{p}, \ \overrightarrow{IB} = \vec{q} \text{ and } \overrightarrow{IC} = \vec{r}.$$

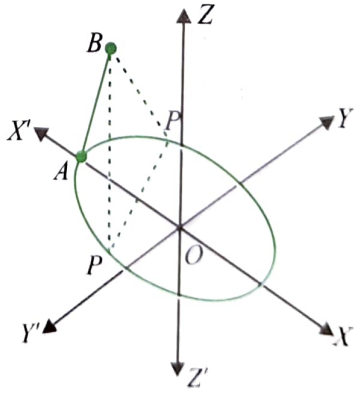
Incentre I is $\frac{a\vec{p}+b\vec{q}+c\vec{r}}{a+b+c}$, where $p = BC$, $q = AC$ and $r = AB$

Incentre is at the origin. Therefore,

$$\frac{a\vec{p}+b\vec{q}+c\vec{r}}{a+b+c} = \vec{0}, \text{ or } a\vec{p}+b\vec{q}+c\vec{r} = \vec{0}$$

$$\Rightarrow \quad a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = \vec{0}$$

29. (1)



Point P lies on $x^2 + 3y^2 = 3$ (i)

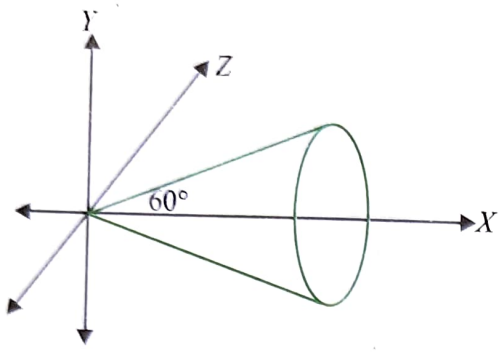
Now from the diagram, according to the given conditions, $AP = AB$

or $(x+\sqrt{3})^2 + (y-0)^2 = 4$ or $(x+\sqrt{3})^2 + y^2 = 4$ (ii)

Solving (i) and (ii), we get $x = 0$ and $y = \pm 1$

Hence, point P has position vector $\pm\hat{j}$

30. (2) From the diagram, it is obvious that locus is a cone concentric with the positive x-axis having vertex at the origin and the slant height equal to the magnitude of the vector.



31. (3) Since $\vec{x}, \vec{y}$ and $\vec{x} \times \vec{y}$ are linearly independent, we have

$$20a - 15b = 15b - 12c = 12c - 20a = 0$$

$$\Rightarrow \quad \frac{a}{3} = \frac{b}{4} = \frac{c}{5}$$

$$\Rightarrow \quad c^2 = a^2 + b^2$$

Hence, ΔABC is right angled.

32. (1) The position vector of any point at t is

$$\vec{r} = (2 + t^2)\hat{i} + (4t - 5)\hat{j} + (2t^2 - 6)\hat{k}$$

$$\Rightarrow \quad \frac{d\vec{r}}{dt} = 2t\,\hat{i} + 4\hat{j} + (4t - 6)\hat{k}$$

$$\Rightarrow \quad \left.\frac{d\vec{r}}{dt}\right|_{t=2} = 4\hat{i} + 4\hat{j} + 2\hat{k}$$

and $$\left.\left|\frac{d\vec{r}}{dt}\right|\right|_{t=2} = \sqrt{16 + 16 + 4} = 4$$

Hence, the required unit tangent vector at $t = 2$ is

$$\frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k}).$$

33. (1) As $\vec{x}$ $\vec{y}$ and $\vec{x} \times \vec{y}$ are non-collinear vectors, vectors are linearly independent. Hence,

$$a - b = 0 = b - c = c - a$$

or $\quad a = b = c$

Therefore, the triangle is equilateral.

34. (3)

Multiple Correct Answers Type

1. (1), (2), (3), (4)

$x\hat{i} + (x+1)\hat{j} + (x+2)\hat{k}, (x+3)\hat{i} + (x+4)\hat{j} + (x+5)\hat{k}$ and $(x+6)\hat{i} + (x+7)\hat{j} + (x+8)\hat{k}$ are coplanar. We have determinant of their coefficients as

$$\begin{vmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have

$$\begin{vmatrix} x & 1 & 2 \\ x+3 & 1 & 2 \\ x+6 & 1 & 2 \end{vmatrix} = 0$$

Hence, $x \in R$.

2. (1), (4)

Let $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Then the diagonals of the parallelogram are $\vec{p} = \vec{a} + \vec{b}$ and $\vec{q} = \vec{b} - \vec{a}$,

i.e., $\vec{p} = 3\hat{i} + 6\hat{j} - 2\hat{k}$, $\vec{q} = -\hat{i} - 2\hat{j} + 8\hat{k}$

So, unit vectors along the diagonals are

$$\frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k}) \text{ and } \frac{1}{\sqrt{69}}(-\hat{i} - 2\hat{j} + 8\hat{k}).$$

3. (2), (3)

We have, $\vec{a} = 2p\,\hat{i} + \hat{j}$

On rotation, let $\vec{b}$ be the vector with components $(p+1)$ and 1 so that $\vec{b} = (p+1)\hat{i} + \hat{j}$.

Now, $|\vec{a}| = |\vec{b}|$ or $a^2 = b^2$

$\Rightarrow \quad 4p^2 + 1 = (p+1)^2 + 1$

or $\quad 4p^2 = (p+1)^2$

or $\quad 2p = \pm(p+1)$

$\Rightarrow \quad 3p = -1$ or $p = 1$

$\therefore \quad p = -1/3$ or $p = 1$

4. (1), (2), (4)

Points $A(\hat{i} + \hat{j})$, $B(\hat{i} - \hat{j})$ and $C(p\hat{i} + q\hat{j} + r\hat{k})$ are collinear

Now $\vec{AB} = -2\hat{j}$ and $\vec{BC} = (p-1)\hat{i} + (q-1)\hat{j} + r\hat{k}$

Vectors $\vec{AB}$ and $\vec{BC}$ must be collinear

$\Rightarrow \quad p = 1, r = 0$ and $q \neq 1$

5. **(1), (2), (3)**

For coplanar vectors, $\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & \mu \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} = 0$

or $(2\lambda - 1)\lambda = 0$ or $\lambda = 0, \frac{1}{2}$

6. **(2), (3)**

Let $\vec{R}$ be the resultant. Then

$$\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3} = (p+1)\hat{i} + 4\hat{j}$$

Given, $|\vec{R}| = 5$. Therefore,

$$(p+1)^2 + 16 = 25$$

or $p + 1 = \pm 3$

or $p = 2, -4$

7. **(1), (2), (4)**

$$\vec{a} = \left[\pm\left(\hat{i} - \hat{j}\right) \pm \left(\hat{j} + \hat{k}\right)\right]$$

$$= \pm\left(\hat{i} + \hat{k}\right), \pm\left(\hat{i} - 2\hat{j} - \hat{k}\right)$$

8. **(2), (3)**

Let $\vec{\alpha} = \hat{i} + x\hat{j} + 3\hat{k}$, $\vec{\beta} = 4\hat{i} + (4x - 2)\hat{j} + 2\hat{k}$

Given, $2\,|\vec{\alpha}| = |\vec{\beta}|$

or $2\sqrt{10 + x^2} = \sqrt{20 + 4(2x - 1)^2}$

or $10 + x^2 = 5 + (4x^2 - 4x + 1)$

or $3x^2 - 4x - 4 = 0$

or $x = 2, -\frac{2}{3}$

9. **(3), (4)**

Let $\vec{a}, \vec{b}$ and $\vec{c}$ lie in the x–y plane.

Let $\vec{a} = \hat{i}$, $\vec{b} = -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{i}$ and $\vec{c} = -\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}$.

Therefore,

$$|\vec{p} + \vec{q} + \vec{r}| = |\lambda\vec{a} + \mu\vec{b} + \nu\vec{c}|$$

$$= \left|\lambda\hat{i} + \mu\left(-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) + \nu\left(-\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}\right)\right|$$

$$= \left|\left(\lambda - \frac{\mu}{2} - \frac{\nu}{2}\right)\hat{i} + \frac{\sqrt{3}}{2}(\mu - \nu)\hat{j}\right|$$

$$= \sqrt{\left(\lambda - \frac{\mu}{2} - \frac{\nu}{2}\right)^2 + \frac{3}{4}(\mu - \nu)^2}$$

$$= \sqrt{\lambda^2 + \mu^2 + \nu^2 - \lambda\mu - \lambda\nu - \mu\nu}$$

$$= \frac{1}{\sqrt{2}}\sqrt{(\lambda - \mu)^2 + (\mu - \nu)^2 + (\nu - \lambda)^2}$$

$$\geq \frac{1}{\sqrt{2}}\sqrt{1 + 1 + 4} = \sqrt{3}$$

Hence, $|\vec{p} + \vec{q} + \vec{r}|$ can take a value equal to $\sqrt{3}$ and 2.

10. **(2), (4)**

Since $\vec{a}$ and $\vec{b}$ are equally inclined to $\vec{c}$, $\vec{c}$ must be of the form $t\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$.

Now $\frac{|\vec{b}|}{|\vec{a}| + |\vec{b}|}\vec{a} + \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}\vec{b}$

$$= \frac{|\vec{a}||\vec{b}|}{|\vec{a}| + |\vec{b}|}\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$$

Also, $\frac{|\vec{b}|}{2|\vec{a}| + |\vec{b}|}\vec{a} + \frac{|\vec{a}|}{2|\vec{a}| + |\vec{b}|}\vec{b}$

$$= \frac{|\vec{a}||\vec{b}|}{2|\vec{a}| + |\vec{b}|}\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$$

Other two vectors cannot be written in the form $t\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$.

11. **(1), (3), (4)**

$$\overrightarrow{OA} = -4\hat{i} + 3\hat{k}\,;\ \overrightarrow{OB} = 14\hat{i} + 2\hat{j} - 5\hat{k}$$

$$\hat{a} = \frac{-4\hat{i} + 3\hat{k}}{5}\,;\ \hat{b} = \frac{14\hat{i} + 2\hat{j} - 5\hat{k}}{15}$$

$$\vec{r} = \frac{\lambda}{15}[-12\hat{i} + 9\hat{j} + 14\hat{i} + 2\hat{j} - 5\hat{k}]$$

$$= \frac{\lambda}{15}[2\hat{i} + 2\hat{j} + 4\hat{k}]$$

$$= \frac{2\lambda}{15}[\hat{i} + \hat{j} + 2\hat{k}]$$

12. **(1), (2), (4)**

$$(\lambda - 1)(\vec{a_1} - \vec{a_2}) + \mu(\vec{a_2} + \vec{a_3}) + \gamma(\vec{a_3} + \vec{a_4} - 2\vec{a_2})$$

$$+ \vec{a_3} + \delta\vec{a_4} = \vec{0}$$

i.e., $(\lambda-1)\vec{a_1}+(1-\lambda+\mu-2\gamma)\vec{a_2}+(\mu+\gamma+1)\vec{a_3}+(\gamma+\delta)\vec{a_4}=\vec{0}$

Since $\vec{a_1}, \vec{a_2}, \vec{a_3}$ and $\vec{a_4}$ are linearly independent, we have

$\lambda-1=0,\ 1-\lambda+\mu-2\gamma=0,\ \mu+\gamma+1=0$ and $\gamma+\delta=0$

i.e., $\lambda=1,\ \mu=2\gamma,\ \mu+\gamma+1=0,\ \gamma+\delta=0$

Hence, $\lambda=1,\ \mu=-\frac{2}{3},\ \gamma=-\frac{1}{3},\ \delta=\frac{1}{3}$

13. (1), (3)

We have, $\overrightarrow{AB}=-\hat{i}-\hat{j}-4\hat{k}$, $\overrightarrow{BC}=-3\hat{i}+3\hat{j}$ and $\overrightarrow{CA}=4\hat{i}-2\hat{j}+4\hat{k}$. Therefore,

$|\overrightarrow{AB}|=|\overrightarrow{BC}|=3\sqrt{2}$ and $|\overrightarrow{CA}|=6$

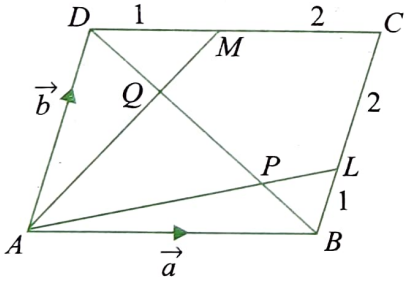
Clearly, $|\overrightarrow{AB}|^2+|\overrightarrow{BC}|^2=|\overrightarrow{AC}|^2$

Hence, the triangle is right-angled isosceles triangle.

Linked Comprehension Type

For Problems 1–3

1. (3), 2. (2), 3. (3)



$\overrightarrow{BL}=\frac{1}{3}\vec{b}$

$\therefore\quad \overrightarrow{AL}=\vec{a}+\frac{1}{3}\vec{b}$

Let $\overrightarrow{AP}=\lambda\overrightarrow{AL}$ and P divides DB in the ratio $\mu:1-\mu$. Then

$\overrightarrow{AP}=\lambda\vec{a}+\frac{\lambda}{3}\vec{b}$ (i)

Also $\overrightarrow{AP}=\mu\vec{a}+(1-\mu)\vec{b}$ (ii)

From (i) and (ii), $\lambda\vec{a}+\frac{\lambda}{3}\vec{b}=\mu\vec{a}+(1-\mu)\vec{b}$

$\therefore\quad \lambda=\mu$ and $\frac{\lambda}{3}=1-\mu$

$\therefore\quad \lambda=\frac{3}{4}$

Hence, P divides AL in the ratio $3:1$ and P divides DB in the ratio $3:1$.

Similarly, Q divides DB in the ratio $1:3$.

Thus, $DQ=\frac{1}{4}DB$ and $PB=\frac{1}{4}DB$

$\therefore\quad PQ=\frac{1}{2}DB$, i.e., $PQ:DB=1:2$

For Problems 4 and 5

4. (3), 5. (2)

Let the position vectors of A, B, C and D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively. Then,

$OA:CB=2:1$

$\Rightarrow\quad \overrightarrow{OA}=2\overrightarrow{CB}$

$\Rightarrow\quad \vec{a}=2(\vec{b}-\vec{c})$ (i)

and $OD:AB=1:3$

$3\overrightarrow{OD}=\overrightarrow{AB}$

$\Rightarrow\quad 3\vec{d}=(\vec{b}-\vec{a})=\vec{b}-2(\vec{b}-\vec{c})$ [using (i)]

$=-\vec{b}+2\vec{c}$ (ii)

Let $OX:XC=\lambda:1$ and $AX:XD=\mu:1$

Now, X divides OC in the ratio $\lambda:1$. Therefore,

P.V. of $X=\frac{\lambda\vec{c}}{\lambda+1}$ (iii)

X also divides AD in the ratio $\mu:1$. Therefore,

P.V. of $X=\frac{\mu\vec{d}+\vec{a}}{\mu+1}$ (iv)

From (iii) and (iv), we get

$\frac{\lambda\vec{c}}{\lambda+1}=\frac{\mu\vec{d}+\vec{a}}{\mu+1}$

or $\left(\frac{\lambda}{\lambda+1}\right)\vec{c}=\left(\frac{\mu}{\mu+1}\right)\vec{d}+\left(\frac{1}{\mu+1}\right)\vec{a}$

or $\left(\frac{\lambda}{\lambda+1}\right)\vec{c}=\left(\frac{\mu}{\mu+1}\right)\left(\frac{-\vec{b}+2\vec{c}}{3}\right)+\left(\frac{1}{\mu+1}\right)2\left(\vec{b}-\vec{c}\right)$ [using (i) and (ii)]

or $\left(\frac{\lambda}{\lambda+1}\right)\vec{c}=\left(\frac{6-\mu}{3(\mu+1)}\right)\vec{b}+\left(\frac{2\mu}{3(\mu+1)}-\frac{2}{\mu+1}\right)\vec{c}$

or $\left(\frac{\lambda}{\lambda+1}\right)\vec{c}=\left(\frac{6-\mu}{3(\mu+1)}\right)\vec{b}+\left(\frac{2\mu-6}{3(\mu+1)}\right)\vec{c}$

or $\left(\frac{6-\mu}{3(\mu+1)}\right)\vec{b}+\left(\frac{2\mu-6}{3(\mu+1)}-\frac{\lambda}{\lambda+1}\right)\vec{c}=\vec{0}$

or $\frac{6-\mu}{3(\mu+1)}=0$ and $\frac{2\mu-6}{3(\mu+1)}-\frac{\lambda}{\lambda+1}=0$

(as $\vec{b}$ and $\vec{c}$ are non-collinear)

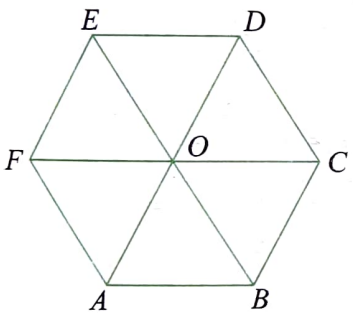
or $\mu=6, \lambda=\frac{2}{5}$

Hence, $OX : XC = 2 : 5$

For Problems 6 and 7

6. (3), 7. (4)

Consider the regular hexagon $ABCDEF$ with centre at O (origin).



$$\overrightarrow{AD}+\overrightarrow{EB}+\overrightarrow{FC}=2\overrightarrow{AO}+2\overrightarrow{OB}+2\overrightarrow{OC}$$

$$=2(\overrightarrow{AO}+\overrightarrow{OB})+2\overrightarrow{OC}$$

$$=2\overrightarrow{AB}+2\overrightarrow{AB}$$

$$(\because \overrightarrow{OC}=\overrightarrow{AB})$$

$$=4\overrightarrow{AB}$$

$$\vec{R}=\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}$$

$$=\overrightarrow{ED}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{CD}$$

$$(\because \overrightarrow{AB}=\overrightarrow{ED} \text{ and } \overrightarrow{AF}=\overrightarrow{CD})$$

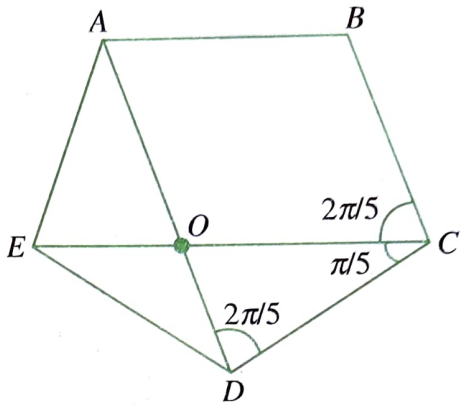
$$=(\overrightarrow{AC}+\overrightarrow{CD})+(\overrightarrow{AE}+\overrightarrow{ED})+\overrightarrow{AD}$$

$$=\overrightarrow{AD}+\overrightarrow{AD}+\overrightarrow{AD}=3\overrightarrow{AD}=6\overrightarrow{AO}$$

For Problems 8 and 9

8. (3), 9. (3)

Given $ABCDE$ is a regular pentagon



Let position vector of point A and C be $\vec{a}$ and $\vec{b}$, respectively.

AD is parallel to BC and AB is parallel to EC. Therefore,

$AOCB$ is a parallelogram

and position vector of B is $\vec{a}+\vec{b}$

The position vectors of E and D are $\lambda\vec{b}$ and $\lambda\vec{a}$ respectively.

Also $OA = BC = AB = OC = 1$ (let)

Therefore, $AOCB$ is rhombus.

$$\angle ABC = \angle AOC = \frac{3\pi}{5}$$

and $$\angle OAB = \angle BCO = \pi - \frac{3\pi}{5} = \frac{2\pi}{5}$$

Further $OA = AE = 1$ and $OC = CD = 1$

Thus, ΔEAO and ΔOCD are isosceles.

In ΔOCD, using sine rule we get $\frac{OC}{\sin\frac{2\pi}{5}}=\frac{OD}{\sin\frac{\pi}{5}}$

$$\Rightarrow \quad OD=\frac{1}{2\cos\frac{\pi}{5}}=OE$$

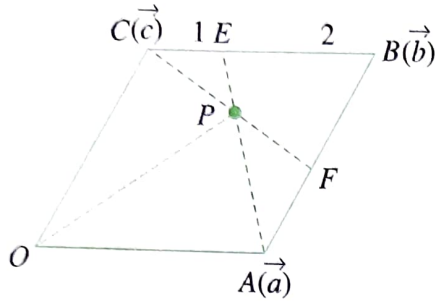
$$\Rightarrow \quad AD=OA+OD=1+\frac{1}{2\cos\frac{\pi}{5}}$$

$$\Rightarrow \quad \frac{AD}{BC}=1+\frac{1}{2\cos\frac{\pi}{5}}=\frac{1+2\cos\frac{\pi}{5}}{2\cos\frac{\pi}{5}}$$

And $$\frac{OE}{OC}=\frac{1}{2\cos\frac{\pi}{5}}$$

For Problems 10 and 11

10. (2), 11. (4)



Let the position vector of A and C be $\vec{a}$ and $\vec{c}$ respectively. Therefore,

Position vector of $B = \vec{b} = \vec{a} + \vec{c}$ (i)

Also Position vector of $E = \dfrac{\vec{b} + 2\vec{c}}{3} = \dfrac{\vec{a} + 3\vec{c}}{3}$ (ii)

Now point P lies on angle bisector of $\angle AOC$. Thus,

Position vector of point $P = \lambda\left(\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}\right)$ (iii)

Also let P divides EA in ration $\mu : 1$. Therefore,
Position vector of P

$$= \frac{\mu\vec{a} + \dfrac{\vec{a} + 3\vec{c}}{3}}{\mu + 1} = \frac{(3\mu + 1)\vec{a} + 3\vec{c}}{3(\mu + 1)} \quad \text{(iv)}$$

Comparing (iii) and (iv), we get

$$\lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right) = \frac{(3\mu + 1)\vec{a} + 3\vec{c}}{3(\mu + 1)}$$

$$\Rightarrow \quad \frac{\lambda}{|\vec{a}|} = \frac{3\mu + 1}{3(\mu + 1)} \text{ and } \frac{\lambda}{|\vec{c}|} = \frac{1}{\mu + 1}$$

$$\Rightarrow \quad \frac{3|\vec{c}| - |\vec{a}|}{3|\vec{a}|} = \mu$$

$$\Rightarrow \quad \frac{\lambda}{|\vec{c}|} = \frac{1}{\dfrac{3|\vec{c}| - |\vec{a}|}{3|\vec{a}|} + 1}$$

$$\Rightarrow \quad \lambda = \frac{3|\vec{a}||\vec{c}|}{3|\vec{c}| + 2|\vec{a}|}$$

Hence, position vector of P is $\dfrac{3|\vec{a}||\vec{c}|}{3|\vec{c}| + 2|\vec{a}|}\left(\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{c}}{|\vec{c}|}\right)$

Let F divides AB in ratio $t : 1$, then position vector of F is $\dfrac{t\vec{b} + \vec{a}}{t + 1}$.

Now points C, P, F are collinear. Then,

$$\overrightarrow{CF} = m\overrightarrow{CP}$$

$$\Rightarrow \quad \frac{t(\vec{a} + \vec{c})}{t + 1} - \vec{c} = m\left\{\frac{3|\vec{a}||\vec{c}|}{3|\vec{c}| + 2|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right) - \vec{c}\right\}$$

Comparing coefficients we get

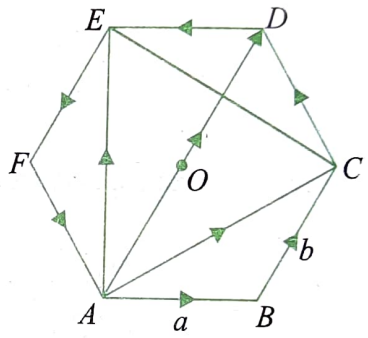
$$\frac{t}{t + 1} = m\frac{3|\vec{c}|}{3|\vec{c}| + 2|\vec{a}|} \text{ and } \frac{-1}{t + 1} = m\frac{|\vec{a}| - 3|\vec{c}|}{3|\vec{c}| + 2|\vec{a}|}$$

$$t = \frac{3|\vec{c}|}{3|\vec{c}| - |\vec{a}|}$$

Matrix Match Type

1. a → p, r, s; b → q, r, s; c → p, r; d → r, s

2. a → q, r; b → p, r; c → q, s; d → p



$$\overrightarrow{AB} = \vec{a}, \ \overrightarrow{BC} = \vec{b}$$

$$\therefore \quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{a} \quad \text{(i)}$$

$$\overrightarrow{AD} = 2\overrightarrow{BC} = 2\vec{b} \quad \text{(ii)}$$

(because AD is parallel to BC and twice its length).

$$\overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC} = 2\vec{b} - (\vec{a} + \vec{b})$$

$$= \vec{b} - \vec{a}$$

$$\overrightarrow{FA} = -\overrightarrow{CD} = \vec{a} - \vec{b} \quad \text{(iii)}$$

$$\overrightarrow{DE} = -\overrightarrow{AB} = -\vec{a} \quad \text{(iv)}$$

$$\overrightarrow{EF} = -\overrightarrow{BC} = -\vec{b} \quad \text{(v)}$$

$$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE} = 2\vec{b} - \vec{a} \quad \text{(vi)}$$

$$\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{DE} = \vec{b} - \vec{a} - \vec{a} = \vec{b} - 2\vec{a} \quad \text{(vii)}$$

Numerical Value Type

1. (2) L.H.S. $= \vec{d} - \vec{a} + \vec{d} - \vec{b} + \vec{h} - \vec{c} + 3(\vec{g} - \vec{h})$

$$= 2\vec{d} - (\vec{a} + \vec{b} + \vec{c}) + 3\frac{(\vec{a} + \vec{b} + \vec{c})}{3} - 2\vec{h}$$

$$= 2\vec{d} - 2\vec{h} = 2(\vec{d} - \vec{h}) = 2\vec{HD}$$

$$\Rightarrow \quad \lambda = 2$$

2. (6) Let $\vec{R}$ be the resultant. Then

$$\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3} = (p+1)\hat{i} + 4\hat{j}$$

Given, $|\vec{R}| = 5$, therefore $R^2 = 25$

$\because \quad (p+1)^2 + 16 = 25$ or $p + 1 = \pm 3$

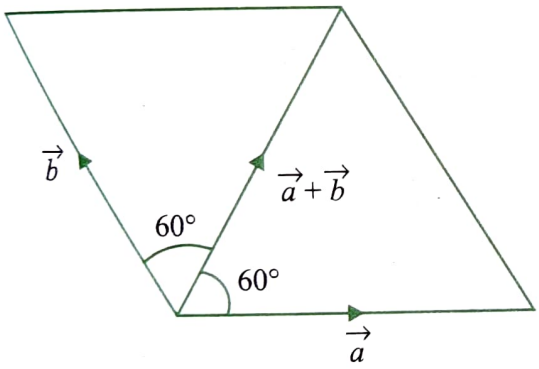
or $\quad p = 2, -4$

3. (3) Given, $\vec{a} + \vec{b} = \vec{c}$

Now vector $\vec{c}$ is along the diagonal of the parallelogram which has adjacent side vectors $\vec{a}$ and $\vec{b}$.

Since $\vec{c}$ is also a unit vector, triangle formed by vectors $\vec{a}$ and $\vec{b}$ is an equilateral triangle. Then

Area of triangle $= \dfrac{\sqrt{3}}{4}$



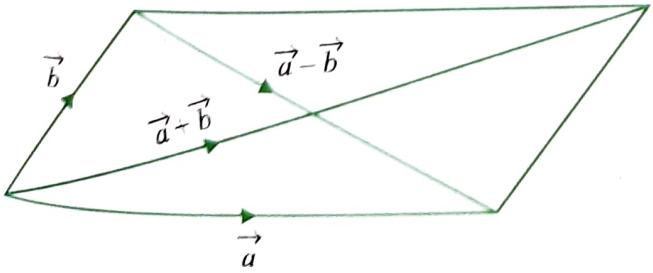
4. (2) Let $\vec{a} = x\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2x\hat{i} + x\hat{j} - \hat{k}$ be the adjacent sides of the parallelogram.

Now angle between $\vec{a}$ and $\vec{b}$ is acute, i.e.,

$$|\vec{a} + \vec{b}| > |\vec{a} - \vec{b}|$$

$$\Rightarrow \quad \left|3x\hat{i} + (x-3)\hat{j} - 2\hat{k}\right|^2 > \left|-x\hat{i} - (x+3)\hat{j}\right|^2$$

or $\quad 9x^2 + (x-3)^2 + 4 > x^2 + (x+3)^2$



or $\quad 8x^2 - 12x + 4 > 0$

or $\quad 2x^2 - 3x + 1 > 0$

or $\quad (2x-1)(x-1) > 0$

$\Rightarrow \quad x < 1/2$ or $x > 1$

Hence, the least positive integral value is 2.

5. (7) Vectors along the sides are $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$

Clearly the vector along the longer diagonal is

$$\vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} + 2\hat{k}$$

Hence, length of the longer diagonal is

$$|\vec{a} + \vec{b}| = |3\hat{i} + 6\hat{j} + 2\hat{k}| = 7$$

6. (9) Vector $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \lambda\hat{i} + \hat{j} + 2\hat{k}$ are coplanar

$$\Rightarrow \quad \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ \lambda & 1 & 2 \end{vmatrix} = 0$$

or $\quad \lambda - 3 + 2(-5) = 0$

or $\quad \lambda = 13$

Archives

JEE Main

Single Correct Answer Type

1. (3) Median through 'A' $= \dfrac{(3\hat{i} + 3\hat{k}) + (5\hat{i} - 2\hat{j} + 4\hat{k})}{2}$

$$= 4\hat{i} - \hat{j} + 4\hat{k}$$

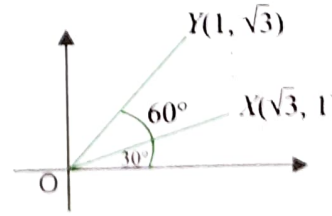
So, required length $= \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$ units.

JEE Advanced

Matrix Match Type

1. **c → p, q**

We have $\vec{OX} = \sqrt{3}\hat{i} + \hat{j}$ and $\vec{OY} = \hat{i} + \sqrt{3}\hat{j}$



Hence, equation of acute angle bisector of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is

$$y = x$$

or $$x - y = 0$$

Now, distance of $\beta\hat{i} + (1-\beta)\hat{i} \equiv Z$ or $(\beta, 1-\beta)$ from $x - y = 0$, is

$$\Rightarrow \quad \left|\frac{\beta - (1-\beta)}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$$

$$\Rightarrow \quad |2\beta - 1| = 3$$

$$\Rightarrow \quad 2\beta - 1 = \pm 3$$

$$\Rightarrow \quad 2\beta = 4, -2$$

$$\Rightarrow \quad \beta = 2, -1$$

$$\Rightarrow \quad |\beta| = 2, 1$$

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (5) Let (1, 1, 1), (–1, 1, 1), (1, –1, 1), (–1, –1, 1) be vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$, rest of the vectors are $-\vec{a}, -\vec{b}, -\vec{c}, -\vec{d}$ and let us find the number of ways of selecting co–planar vectors.

Observe that out of any three coplanar vectors two will be collinear (anti parallel).

Number of ways of selecting the anti-parallel pair = 4

Number of ways of selecting the third vector = 6

Total = 24

Number of non-coplanar selections

$$= {}^8C_3 - 24 = 32 = 2^5, p = 5$$

$$\therefore \quad p = 5$$

2. (9) According to question $\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$ and

$$\vec{s} = x(-\vec{p} + \vec{q} + \vec{r}) + y(\vec{p} - \vec{q} + \vec{r}) + z(-\vec{p} - \vec{q} + \vec{r})$$

$$\therefore \quad -x + y - z = 4 \quad (1)$$

$$x - y - z = 3 \quad (2)$$

$$x + y + z = 5 \quad (3)$$

Adding (1) and (2), we get

$$z = -\frac{7}{2}$$

Adding (2) and (3), we get

$$x = 4$$

Adding (1) and (3), we get

$$y = 9/2$$

Concept Application Exercises

Exercise 3.1

1. $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$

or $\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=8$

or $|\vec{a}|^2-|\vec{b}|^2=8$

or $(8|\vec{b}|)^2-|\vec{b}|^2=8 \qquad [\because |\vec{a}|=8|\vec{b}|]$

or $|\vec{b}|=\sqrt{\dfrac{8}{63}}$

or $|\vec{a}|=8|\vec{b}|=\dfrac{8\times 2\sqrt{2}}{3\sqrt{7}}=\dfrac{16\sqrt{2}}{3\sqrt{7}}$

2. $(|\vec{a}|\vec{b}+|\vec{b}|\vec{a})\cdot(|\vec{a}|\vec{b}-|\vec{b}|\vec{a})$

$=|\vec{a}|^2\vec{b}\cdot\vec{b}-|\vec{a}||\vec{b}|\vec{b}\cdot\vec{a}$
$+|\vec{b}||\vec{a}|\vec{a}\cdot\vec{b}-|\vec{b}|^2\vec{a}\cdot\vec{a}$
$=|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2$
$=0$

Hence, $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$ are perpendicular to each other.

3. The vertices of ΔABC are given as $A(1, 2, 3)$, $B(-1, 0, 0)$, and $C(0, 1, 2)$.
Also, it is given that $\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Thus,

$\overrightarrow{BA}=\{1-(-1)\}\hat{i}+(2-0)\hat{j}+(3-0)\hat{k}$
$=2\hat{i}+2\hat{j}+3\hat{k}$
$\overrightarrow{BC}=\{0-(-1)\}\hat{i}+(1-0)\hat{j}+(2-0)\hat{k}$
$=\hat{i}+\hat{j}+2\hat{k}$
$\therefore \quad \overrightarrow{BA}\cdot\overrightarrow{BC}=(2\hat{i}+2\hat{j}+3\hat{k})\cdot(\hat{i}+\hat{j}+2\hat{k})$
$=2\times 1+2\times 1+3\times 2$
$=2+2+6=10$
$|\overrightarrow{BA}|=\sqrt{2^2+2^2+3^2}=\sqrt{4+4+9}=\sqrt{17}$
$|\overrightarrow{BC}|=\sqrt{1+1+2^2}=\sqrt{6}$

Now, it is known that:

$\overrightarrow{BA}\cdot\overrightarrow{BC}=|\overrightarrow{BA}||\overrightarrow{BC}|\cos(\angle ABC)$

$\therefore \quad 10=\sqrt{17}\times\sqrt{6}\cos(\angle ABC)$

or $\cos(\angle ABC)=\dfrac{10}{\sqrt{17}\times\sqrt{6}}$

or $\angle ABC=\cos^{-1}\left(\dfrac{10}{\sqrt{102}}\right)$

4. $|4\vec{a}+3\vec{b}|=\sqrt{(4\vec{a}+3\vec{b})\cdot(4\vec{a}+3\vec{b})}$

$=\sqrt{16|\vec{a}|^2+9|\vec{b}|^2+24\vec{a}\cdot\vec{b}}$
$=\sqrt{144+144+24\times 3\times 4\times\left(\dfrac{-1}{2}\right)}$
$=12$

5. It is given that vectors $\hat{i}-2x\hat{j}-3y\hat{k}$ and $\hat{i}+3x\hat{j}+2y\hat{k}$ are orthogonal. Therefore,

$(\hat{i}-2x\hat{j}-3y\hat{k})\cdot(\hat{i}+3x\hat{j}+2y\hat{k})=0$

or $1-6x^2-6y^2=0$

or $6x^2+6y^2=1$, which is a circle.

6. $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$
$=1+4+4+0+0+0=9$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|=3$

7. Given, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\vec{a}+\vec{b}=-\vec{c}$
$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(-\vec{c})\cdot(-\vec{c})$

or $a^2+b^2+2(\vec{a}\cdot\vec{b})=c^2$

or $9+25+2(\vec{a}\cdot\vec{b})=49$

or $\vec{a}\cdot\vec{b}=15/2$

or $ab\cos\theta=15/2$ or $3\cdot 5\cos\theta=15/2$

or $\cos\theta=1/2\Rightarrow\theta=\pi/3$

8. $|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$=a^2+b^2-2(\vec{a}\cdot\vec{b})$
$1+1-2(1\cdot 1\cdot\cos\theta)$
$=2(1-\cos\theta)$
$=2\left(1-\dfrac{1}{2}\right)=1$

9. $\hat{n} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where $a_1^2 + a_2^2 + a_3^2 = 1$

Given that $\vec{u}\cdot\hat{n} = 0 \Rightarrow a_1 + a_2 = 0$

Also, $\vec{v}\cdot\hat{n} = 0 \Rightarrow a_1 - a_2 = 0$

$a_1 = a_2 = 0$

$a_3 = 1$ or -1

$\hat{n} = \hat{k}$ or $-\hat{k}$

$|\vec{w}\cdot\hat{n}| = 3$

10. $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \vec{a} + \vec{b} + \vec{c}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$ or $\overrightarrow{CA} = -(\vec{a} + \vec{b})$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \vec{b} + \vec{c}$

$\therefore \quad \overrightarrow{AB}\cdot\overrightarrow{CD} + \overrightarrow{BC}\cdot\overrightarrow{AD} + \overrightarrow{CA}\cdot\overrightarrow{BD}$

$= \vec{a}\cdot\vec{c} + \vec{b}\cdot(\vec{a} + \vec{b} + \vec{c}) + (-\vec{a} - \vec{b})\cdot(\vec{b} + \vec{c})$

$= \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{c} - \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c}$

$-\vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{c} = 0$

11. $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \hat{i} - 2\hat{k}$

$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = 4\hat{i} - 4\hat{j} - \hat{k}$

Projection of $\overrightarrow{PQ}$ on $\overrightarrow{RS} = \dfrac{\overrightarrow{QP}\cdot\overrightarrow{RS}}{|\overrightarrow{RS}|} = \dfrac{6}{\sqrt{33}}$

12. $3\vec{p} + \vec{q}$ and $5\vec{p} - 3\vec{q}$ are perpendicular. Therefore,

$(3\vec{p} + \vec{q})\cdot(5\vec{p} - 3\vec{q}) = 0$

$15\vec{p}^2 - 3\vec{q}^2 = 4\vec{p}\cdot\vec{q}$ (i)

$2\vec{p} + \vec{q}$ and $4\vec{p} - 2\vec{q}$ are perpendicular. Therefore,

$(2\vec{p} + \vec{q})\cdot(4\vec{p} - 2\vec{q}) = 0$

$8\vec{p}^2 = 2\vec{q}^2$

$\vec{q}^2 = 4\vec{p}^2$ (ii)

Now, $\cos\theta = \dfrac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|}$

Substituting $\vec{q}^2 = 4\vec{p}^2$ in (i), $3\vec{p}^2 = 4\vec{p}\cdot\vec{q}$

$\therefore \quad \cos\theta = \dfrac{3}{4}\cdot\dfrac{\vec{p}^2}{|\vec{p}|2|\vec{p}|} = \dfrac{3}{8}$

or $\theta = \cos^{-1}\dfrac{3}{8}$

13. $\vec{A}\cdot(\alpha\vec{A} + \vec{B}) = \vec{B}\cdot(\alpha\vec{A} + \vec{B})$

or $\alpha + \vec{A}\cdot\vec{B} = \alpha\vec{A}\cdot\vec{B} + 1$

or $(\vec{A}\cdot\vec{B})(1 - \alpha) = (1 - \alpha)$

Since $\vec{A}\cdot\vec{B} \neq 0 \Rightarrow \alpha = 1$

14. $\vec{a} + \vec{b} + \vec{c} = \vec{x}$

Taking dot with $\vec{x}$ on both sides, we get

$\vec{x}\cdot\vec{a} + \vec{x}\cdot\vec{b} + \vec{x}\cdot\vec{c} = \vec{x}\cdot\vec{x} = |\vec{x}|^2 = 4$

or $1 + \dfrac{3}{2} + \vec{x}\cdot\vec{c} = 4$ or $\vec{x}\cdot\vec{c} = \dfrac{3}{2}$

If θ is the angle between $\vec{c}$ and $\vec{x}$, then

$|\vec{x}||\vec{c}|\cos\theta = 3/2$

$\Rightarrow \quad \cos\theta = 3/4$ or $\theta = \cos^{-1}(3/4)$

15. Let θ be an angle between unit vectors $\vec{a} + \vec{b}$. Then

$\vec{a}\cdot\vec{b} = \cos\theta$

$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$

$= 2 + 2\cos\theta = 4\cos^2\theta/2$

or $|\vec{a} + \vec{b}| = 2\cos\dfrac{\theta}{2}$

Similarly, $|\vec{a} - \vec{b}| = 2\sin\dfrac{\theta}{2}$

$\Rightarrow \quad |\vec{a} + \vec{b}| + |\vec{a} - \vec{b}| = 2\left(\cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}\right) \leq 2\sqrt{2}$

16. Resultant force

$\vec{F} = \vec{P_1} + \vec{P_2} + \vec{P_3} = 2\hat{j} - \hat{k}$

And displacement $= \vec{AB}$

$=$ P.V. of B $-$ P.V. of A

$= (6\hat{i} + \hat{j} - 3\hat{k}) - (4\hat{i} - 3\hat{j} - 2\hat{k})$

$= 2\hat{i} + 4\hat{j} - \hat{k}$

$\therefore \quad$ Work done $= \vec{F}\cdot\vec{AB}$

$= (2\hat{j} - \hat{k})\cdot(2\hat{i} + 4\hat{j} - \hat{k})$

$= 9$ units

17. $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$

$\Rightarrow \quad 100 + 64 = 2(25 + |\vec{b}|^2)$

or $82 = 25 + |\vec{b}|^2$

or $|\vec{b}|^2 = 57$

18. Let A be the origin, and the position vectors of B, C and D be $\vec{b}, \vec{c}, \vec{d}$.

$$\overrightarrow{AB}\cdot\overrightarrow{CD} = k(|\overrightarrow{AD}|^2 + |\overrightarrow{BC}|^2 - |\overrightarrow{AC}|^2 - |\overrightarrow{BD}|^2)$$

or $(\vec{b})\cdot(\vec{d}-\vec{c})$
$= k[(\vec{d})^2 + (\vec{c}-\vec{b})^2 - (\vec{c})^2 - (\vec{d}-\vec{b})^2]$

or $\vec{b}\cdot\vec{d} - \vec{b}\cdot\vec{c} = k(-2\vec{b}\cdot\vec{c} + 2\vec{b}\cdot\vec{d})$

or $k = 1/2$

Exercise 3.2

1. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -5 \\ m & n & 12 \end{vmatrix}$

$= (36+5n)\hat{i} - (24+5m)\hat{j} + (2n-3m)\hat{k}$

$= \vec{0}$

$m = \dfrac{-24}{5}, n = \dfrac{-36}{5}$

2. $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, but $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

or $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \dfrac{4}{5} \Rightarrow \cos\theta = \dfrac{3}{5}$

Therefore, $\vec{a}\cdot\vec{b} = 2\times 5\times\dfrac{3}{5} = 6$

3. Since $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} \neq \vec{0}$, we have

$\vec{a}\times\vec{b} - \vec{b}\times\vec{c} = \vec{0}$

or $\vec{a}\times\vec{b} + \vec{c}\times\vec{b} = \vec{0}$

or $(\vec{a}+\vec{c})\times\vec{b} = \vec{0}$

Hence, $\vec{a}+\vec{c}$ is parallel to $\vec{b}$. Thus,

$\vec{a}+\vec{c} = k\vec{b}$

4. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = -10\hat{i} + 9\hat{j} + 7\hat{k}$

$\vec{a}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4\hat{i} - 3\hat{j} - \hat{k}$

$\Rightarrow (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c}) = -40 - 27 - 7 = -74$

5. Since $\vec{a}, \vec{c}$ and $\vec{b}$ form a right-handed system,

$\vec{c} = \vec{b}\times\vec{a}$

$= \hat{j}\times(x\hat{i} + y\hat{j} + z\hat{k})$

$= -x\hat{k} + z\hat{i} = z\hat{i} - x\hat{k}$

6. We have $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$. Therefore,

$\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = 0$ or $\vec{a}\cdot(\vec{b}-\vec{c}) = 0$

Therefore, there are three possibilities: (i) $\vec{a} = \vec{0}$, (ii) $\vec{b} - \vec{c} = \vec{0}$ and (iii) $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$.

Again, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$. Therefore,

$\vec{a}\times\vec{b} - \vec{a}\times\vec{c} = \vec{0}$

or $\vec{a}\times(\vec{b}-\vec{c}) = \vec{0}$

Therefore, again there are three possibilities: (i) $\vec{a} = \vec{0}$, (ii) $\vec{b} - \vec{c} = \vec{0}$ and (iii) $\vec{a}$ is parallel to $\vec{b} - \vec{c}$.

Now $\vec{a}$ is given to be a non-zero vector. Therefore, we have the following possibilities left:

(i) $\vec{b} - \vec{c} = \vec{0}$.

(ii) $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$ and $\vec{a}$ is parallel to $\vec{b} - \vec{c}$, which is absurd.

Therefore, the only possibility left is $\vec{b} - \vec{c} = \vec{0}$ or $\vec{b} = \vec{c}$.

7. $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$

$= \vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b}$

$= \vec{a}\times\vec{a} + \vec{a}\times\vec{b} + \vec{a}\times\vec{b} - \vec{b}\times\vec{b}$

$= \vec{0} + 2\vec{a}\times\vec{b} - \vec{0} = 2\vec{a}\times\vec{b}$

Geometrically, the vector area of a parallelogram whose sides are along vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\times\vec{b}$.

Also diagonals are along vectors $\vec{a}-\vec{b}$ and $\vec{a}+\vec{b}$ and the vector area in terms of diagonal vectors is $\dfrac{1}{2}[(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})]$.

8. $\vec{z} + \vec{z}\times\vec{x} = \vec{y}$ or $|\vec{z} + \vec{z}\times\vec{x}|^2 = |\vec{y}|^2$

or $|\vec{z}|^2 + |\vec{z}|^2|\vec{x}|^2\sin^2\theta = 1$

(because $\vec{z}\cdot(\vec{z}\times\vec{x}) = 0$)

or $|\vec{z}|^2(1+\sin^2\theta)=1$

or $|\vec{z}|=\dfrac{1}{\sqrt{1+\sin^2\theta}}=\dfrac{2}{\sqrt{7}}$

or $\sin\theta = \sqrt{3}/2$

$\Rightarrow$ $\theta=\pi/3=60^\circ$

9. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Therefore,

$\vec{a}\cdot\hat{i}=a_1$, $\vec{a}\cdot\hat{j}=a_2$ and $\vec{a}\cdot\hat{k}=a_3$

and $\vec{a}\times\hat{i}=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{i}=-a_2\hat{k}+a_3\hat{j}$

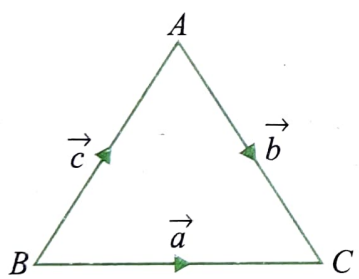
Similarly, $\vec{a}\times\hat{j}=a_1\hat{k}-a_3\hat{i}$ and $\vec{a}\times\hat{k}=-a_1\hat{j}+a_2\hat{i}$

$(\vec{a}\cdot\hat{i})(\vec{a}\times\hat{i})+(\vec{a}\cdot\hat{j})(\vec{a}\times\hat{j})+(\vec{a}\cdot\hat{k})(\vec{a}\times\hat{k})$

$= -a_1a_2\hat{k}+a_1a_3\hat{j}+a_1a_2\hat{k}-a_3a_2\hat{i}$
$+a_3a_2\hat{i}-a_3a_1\hat{j}$

$=\vec{0}$

10.



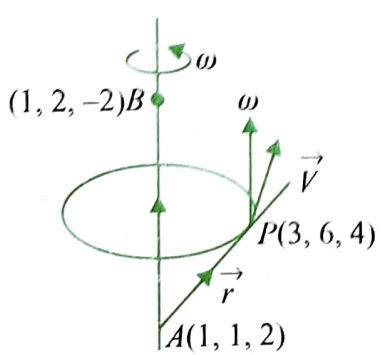
Clearly, $\vec{a}$, $\vec{b}$ and $\vec{c}$ represent the sides of a triangle.

$\Rightarrow$ $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

or $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=3\vec{a}\times\vec{b}$

or $2\vec{b}\times\vec{a}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0 \Rightarrow \lambda=2$

11.



$\overrightarrow{OA}=\hat{i}+\hat{j}+2\hat{k}$

$\overrightarrow{OB}=\hat{i}+2\hat{j}-2\hat{k}$

$\overrightarrow{AB}=\hat{j}-4k \Rightarrow |\vec{AB}|=\sqrt{17}$

$\overrightarrow{AP}=(3\hat{i}+6\hat{j}+4\hat{k})-(\hat{i}+\hat{j}+2\hat{k})$
$=2\hat{i}+5\hat{j}+2\hat{k}$

$\therefore$ $\vec{\omega}=\dfrac{3}{\sqrt{17}}(\hat{j}-4\hat{k})2$

$\vec{v}=\vec{\omega}\times\vec{r}=\dfrac{3}{\sqrt{17}}(\hat{j}-4\hat{k})\times(2\hat{i}+5\hat{j}+2\hat{k})$

$=\dfrac{3}{\sqrt{17}}(22\hat{i}-8\hat{j}-2\hat{k})$

12. We have $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}=0$.

This implies that $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$.

Thus, $\vec{a}$ is a unit vector perpendicular to both $\vec{b}$ and $\vec{c}$. Hence,

$$\vec{a}=\pm\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}=\pm\frac{\vec{b}\times\vec{c}}{|\vec{b}||\vec{c}|\sin\pi/3}$$

$$=\pm2(\vec{b}\times\vec{c})$$

13. Since $(\vec{a}\times\vec{b})^2+(\vec{a}\cdot\vec{b})^2=144$, if the angle between $\vec{a}$ and $\vec{b}$ is θ, then

$|\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta=144$

or $|\vec{a}|^2|\vec{b}|^2=144$

or $|\vec{a}||\vec{b}|=12$

or $4|\vec{b}|=12$

or $|\vec{b}|=3$

14. We have, $|\vec{a}+\vec{b}|=\sqrt{3}$

or $|\vec{a}+\vec{b}|^2=3$

or $|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})=3$

or $1+1+2(\vec{a}\cdot\vec{b})=3$

or $\vec{a}\cdot\vec{b}=1/2$ (i)

Now, $\vec{c}-\vec{a}-2\vec{b}=3(\vec{a}\times\vec{b})$

or $(\vec{c}-\vec{a}-2\vec{b})\cdot\vec{b}=3\{(\vec{a}\times\vec{b})\cdot\vec{b}\}$

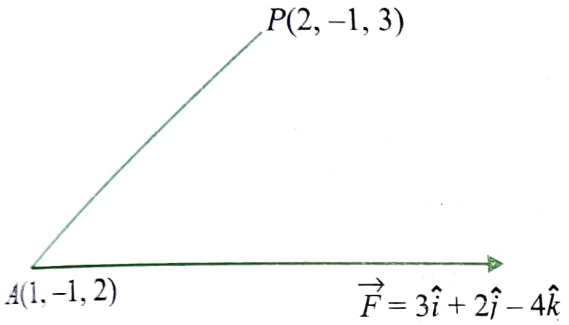
or $\vec{c}\cdot\vec{b}-\vec{a}\cdot\vec{b}-2(\vec{b}\cdot\vec{b})=0$

(because $[\vec{a}\ \vec{b}\ \vec{b}]=0$)

or $\vec{c}\cdot\vec{b}-\frac{1}{2}-2\times1=0$ [Using (i)]

or $\vec{c}\cdot\vec{b}=5/2$

15.



$\vec{F}=3\hat{i}+2\hat{j}-4\hat{k}$

A is $(1,-1,2)$, P is $(2,-1,3)$

$\therefore$ $\vec{PA}$ = P.V. of A – P.V. of P

$=(\hat{i}-\hat{j}+2\hat{k})-(2\hat{i}-\hat{j}+3\hat{k})$

$=-\hat{i}-\hat{k}$

Required vector moment $=\vec{PA}\times\vec{F}$

$=(-\hat{i}-\hat{k})\times(3\hat{i}+2\hat{j}-4\hat{k})$

$$=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\-1&0&-1\\3&2&-4\end{vmatrix}$$

$=2\hat{i}-7\hat{j}-2\hat{k}$

Exercise 3.3

1. Since $\vec{d}$ makes equal angles with the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have

$$d=\frac{\mu(\vec{a}+\vec{b}+\vec{c})}{3} \qquad \text{(i)}$$

($\vec{d}$ passes through the centroid of the triangle with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$)

Again $[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}=[\vec{d}\ \vec{b}\ \vec{c}]\vec{a}+[\vec{d}\ \vec{c}\ \vec{a}]\vec{b}$

$+[\vec{d}\ \vec{a}\ \vec{b}]\vec{c}$ (ii)

From (i) and (ii), we get $[\vec{d}\ \vec{b}\ \vec{c}]=[\vec{d}\ \vec{c}\ \vec{a}]=[\vec{d}\ \vec{a}\ \vec{b}]$

2. Let $\vec{l}=l_1\hat{i}+l_2\hat{j}+l_3\hat{k}$, $\vec{m}=m_1\hat{i}+m_2\hat{j}+m_3\hat{k}$, $\vec{n}=n_1\hat{i}+n_2\hat{j}+n_3\hat{k}$, $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$. Therefore,

$$\vec{l}\cdot\vec{a}=l_1a_1+l_2a_2+l_3a_3=\sum l_1a_1$$

Similarly, $\vec{l}\cdot\vec{b}=\sum l_1b_1$, etc.

Now, $[\vec{l}\ \vec{m}\ \vec{n}](\vec{a}\times\vec{b})$

$$=\begin{vmatrix}l_1&l_2&l_3\\m_1&m_2&m_3\\n_1&n_2&n_3\end{vmatrix}\times\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$$

$$=\begin{vmatrix}\sum l_1\hat{i}&\sum l_1a_1&\sum l_1b_1\\\sum m_1\hat{i}&\sum m_1a_1&\sum m_1b_1\\\sum n_1\hat{i}&\sum n_1a_1&\sum n_1b_1\end{vmatrix}$$

$$=\begin{vmatrix}\vec{l}&\vec{l}\cdot\vec{a}&\vec{l}\cdot\vec{b}\\\vec{m}&\vec{m}\cdot\vec{a}&\vec{m}\cdot\vec{b}\\\vec{n}&\vec{n}\cdot\vec{a}&\vec{n}\cdot\vec{b}\end{vmatrix}=\begin{vmatrix}\vec{l}\cdot\vec{a}&\vec{l}\cdot\vec{b}&\vec{l}\\\vec{m}\cdot\vec{a}&\vec{m}\cdot\vec{b}&\vec{m}\\\vec{n}\cdot\vec{a}&\vec{n}\cdot\vec{b}&\vec{n}\end{vmatrix}$$

3. $\begin{vmatrix}2&3&4\\1&\alpha&2\\1&2&\alpha\end{vmatrix}=15$

or $2(\alpha^2-4)+3(2-\alpha)+4(2-\alpha)=15$

or $2\alpha^2-8+6-3\alpha+8-4\alpha=15$

or $2\alpha^2-7\alpha-9=0$

or $2\alpha^2-9\alpha+2\alpha-9=0$

or $(\alpha+1)(2\alpha-9)=0$

or $\alpha=-1,\ 9/2$

4. $\vec{a}\times\vec{b}=\vec{a}\times(\vec{a}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{a}-(\vec{a}\cdot\vec{a})\vec{c}$

$=2\vec{a}-3\vec{c}$

But $\vec{a}\times\vec{b}=1\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&1&1\\1&-2&1\end{vmatrix}=3\hat{i}-3\hat{k}$

Hence, $3\vec{c}=2\vec{a}-(3\hat{i}-3\hat{k})$

$=(2\hat{i}+2\hat{j}+2\hat{k})-(3\hat{i}-3\hat{k})$

$=-\hat{i}+2\hat{j}+5\hat{k}$

or $\vec{c}=\frac{1}{3}(-\hat{i}+2\hat{j}+5\hat{k})$

5. Since $\vec{x}$ is a non-zero vector, the given conditions will be satisfied if either (i) vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are zero or (ii) $\vec{x}$ is perpendicular to vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

In case (ii) $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar and so $[\vec{a}\,\vec{b}\,\vec{c}]=0$.

6. Here, $\vec{a}\times\vec{b}=\vec{c}$ (given)

$(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{c}\cdot\vec{c}$

$[\vec{a}\,\vec{b}\,\vec{c}]=|\vec{c}|^2$ (i)

Also, $\vec{b}\times\vec{c}=\vec{a}$ (given)

$\Rightarrow\quad(\vec{b}\times\vec{c})\cdot\vec{a}=\vec{a}\cdot\vec{a}$

$\Rightarrow\quad[\vec{b}\,\vec{c}\,\vec{a}]=|\vec{a}|^2$

also $\vec{c}\times\vec{a}=\vec{b}$ (given) (ii)

$(\vec{c}\times\vec{a})\cdot\vec{b}=\vec{b}\cdot\vec{b}$

$[\vec{c}\,\vec{a}\,\vec{b}]=|\vec{b}|^2$

Since $[\vec{a}\,\vec{b}\,\vec{c}]=[\vec{b}\,\vec{c}\,\vec{a}]=[\vec{c}\,\vec{a}\,\vec{b}]$, (iii)

$|\vec{a}|=|\vec{b}|=|\vec{c}|$

7. $\vec{a}=\vec{p}+\vec{q}$

or $\vec{a}\times\vec{b}=\vec{p}\times\vec{b}+\vec{q}\times\vec{b}$

or $\vec{a}\times\vec{b}=\vec{q}\times\vec{b}\qquad(\because\ \vec{p}\times\vec{b}=\vec{0})$

or $\vec{b}\times(\vec{a}\times\vec{b})=\vec{b}\times(\vec{q}\times\vec{b})$

$=(\vec{b}\cdot\vec{b})\vec{q}-(\vec{b}\cdot\vec{q})\vec{b}$

$=(\vec{b}\cdot\vec{b})\vec{q}\qquad(\because\ \vec{b}\cdot\vec{q}=0)$

or $\frac{\vec{b}\times(\vec{a}\times\vec{b})}{\vec{b}\cdot\vec{b}}=\vec{q}$

8. $\vec{a}\cdot(\vec{b}\times\hat{i})\,\hat{i}=((\vec{a}\times\vec{b})\cdot\hat{i})\,\hat{i}$

If $\vec{a}\times\vec{b}=x\hat{i}+y\hat{j}+z\hat{k}$, then $(\vec{a}\times\vec{b})\cdot\hat{i}=x$

Similarly, $(\vec{a}\cdot(\vec{b}\times\hat{j}))\,\hat{j}=y$ and $(\vec{a}\cdot(\vec{b}\times\hat{k}))\,\hat{k}=z$

$\Rightarrow\quad(\vec{a}\cdot(\vec{b}\times\hat{i}))\,\hat{i}+(\vec{a}\cdot(\vec{b}\times\hat{j}))\,\hat{j}+(\vec{a}\cdot(\vec{b}\times\hat{k}))\,\hat{k}$

$=x\hat{i}+y\hat{j}+z\hat{k}=\vec{a}\times\vec{b}$

9. $\vec{d}\cdot(\vec{a}\times(\vec{b}\times(\vec{c}\times\vec{d})))$

$=\vec{d}\cdot(\vec{a}\times((\vec{b}\cdot\vec{d})\vec{c}-(\vec{b}\cdot\vec{c})\vec{d}))$

$=\vec{d}\cdot((\vec{b}\cdot\vec{d})(\vec{a}\times\vec{c})-(\vec{b}\cdot\vec{c})(\vec{a}\times\vec{d}))$

$=(\vec{b}\cdot\vec{d})(\vec{d}\cdot(\vec{a}\times\vec{c}))-(\vec{b}\cdot\vec{c})(\vec{d}\cdot(\vec{a}\times\vec{d}))$

$=(\vec{b}\cdot\vec{d})[\vec{d}\,\vec{a}\,\vec{c}]$

$=(\vec{b}\cdot\vec{d})[\vec{a}\,\vec{c}\,\vec{d}]$

10. $\vec{a}\times(\vec{a}\times\vec{b})=\frac{1}{2}\vec{b}\ \Rightarrow\ (\vec{a}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{a})\vec{b}=\frac{1}{2}\vec{b}$

$\Rightarrow\ \vec{a}\cdot\vec{b}=0,\ \vec{a}\cdot\vec{a}=-\frac{1}{2}$ (which is not possible).

Thus, the given information is inconsistent.

11. We are given that

$(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}.$

$\therefore\quad(\vec{a}\cdot\vec{b})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}=\vec{0}$

$\Rightarrow\quad(\vec{a}\times\vec{c})\times\vec{b}=\vec{0}$ (i)

Thus, either $\vec{b}=\vec{0}$ or $\vec{a}\times\vec{c}=\vec{0}$

But $\vec{b}\neq\vec{0}$, therefore, $\vec{a}\times\vec{c}=\vec{0}$ and hence $\vec{c}=t\vec{a}$,, i.e., $\vec{a}$ and $\vec{c}$ are collinear.

12. We have $(\vec{a}\times\vec{b})\times\vec{c}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$

or $(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$

or $(\vec{a}\cdot\vec{c})\vec{b}-\left\{(\vec{b}\cdot\vec{c})+\frac{1}{3}|\vec{b}||\vec{c}|\right\}\vec{a}=\vec{0}$

$\Rightarrow$ $\vec{a}\cdot\vec{c}=0$ and $\vec{b}\cdot\vec{c}+\frac{1}{3}|\vec{b}||\vec{c}|=0$

($\vec{a}$ and $\vec{b}$ are non-collinear)

or $|\vec{b}||\vec{c}|\cos\theta+\frac{1}{3}|\vec{b}||\vec{c}|=0$

or $\cos\theta=-1/3$

$\Rightarrow$ $\sin\theta=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}$

13. Here $\vec{p}=\vec{b}\times\vec{c},\ \vec{q}=\vec{c}\times\vec{a},\ \vec{r}=\vec{a}\times\vec{b}$

Now $\vec{a}\times(\vec{q}\times\vec{r})=(\vec{a}\cdot\vec{r})\vec{q}-(\vec{a}\cdot\vec{q})\vec{r}$

$=\{\vec{a}\cdot(\vec{a}\times\vec{b})\}\vec{q}-\{\vec{a}\cdot(\vec{c}\times\vec{a})\vec{r}$

$=(0)\vec{q}-(0)\vec{r}=\vec{0}$

Therefore, $\vec{a}$ is parallel to $\vec{q}\times\vec{p}$

Again $\vec{b}\times(\vec{r}\times\vec{p})=(\vec{b}\cdot\vec{p})\vec{r}-(\vec{b}\cdot\vec{r})\vec{p}$

$=\{\vec{b}\cdot(\vec{b}\times\vec{c})\}\vec{r}-\{\vec{b}\cdot(\vec{a}\times\vec{b})\}\vec{p}$

$=(0)\vec{r}-(0)\vec{p}=\vec{0}$

Therefore, $\vec{b}$ is parallel to $\vec{r}\times\vec{p}$

Also $\vec{c}\times(\vec{p}\times\vec{q})=(\vec{c}\cdot\vec{q})\vec{p}-(\vec{c}\cdot\vec{p})\vec{q}$

$=\{\vec{c}\cdot(\vec{c}\times\vec{a})\}\vec{p}-\{\vec{c}\cdot(\vec{b}\times\vec{c})\}\vec{q}$

$=(0)\vec{p}-(0)\vec{q}=\vec{0}$

Therefore, $\vec{c}$ is parallel to $\vec{p}\times\vec{q}$.

14. $\vec{a}\times\vec{a}'=\frac{\vec{a}\times(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Similarly, $\vec{b}\times\vec{b}'=\frac{(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

and $\vec{c}\times\vec{c}'=\frac{(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

So $\vec{a}\times\vec{a}'+\vec{b}\times\vec{b}'+\vec{c}\times\vec{c}'=0$

15. $[\hat{n}\ \hat{p}\ \hat{m}]=[\hat{p}\ \hat{m}\ \hat{n}]$

$=\hat{p}\cdot(\hat{m}\times\hat{n})$

$=|\hat{p}|\cdot|\hat{m}\times\hat{n}|\cos\alpha$

$=|\hat{m}\times\hat{n}|\cos\alpha$

$=\sin\alpha\cos\alpha=1/4$ (given)

$\Rightarrow$ $\sin 2\alpha=1/2$

or $2\alpha=30°$

or $\alpha=15°$

16. $\vec{a}\times\vec{b}=p\vec{a}+q\vec{b}+r\vec{c}$

Taking dot product with $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$$0=p+\frac{3}{5}q+\frac{3}{5}r \quad \text{(i)}$$

$$0=\frac{3}{5}p+q+\frac{3}{5}r \quad \text{(ii)}$$

$$[\vec{a}\,\vec{b}\,\vec{c}]=\frac{3}{5}p+\frac{3}{5}q+r \quad \text{(iii)}$$

Also $[\vec{a}\,\vec{b}\,\vec{c}]^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}$

$=\begin{bmatrix}1 & 3/5 & 3/5\\ 3/5 & 1 & 3/5\\ 3/5 & 3/5 & 1\end{bmatrix}=\frac{44}{125}$

Solving (i), (ii) and (iii) for q, we get

$\frac{5}{11}[\vec{a}\,\vec{b}\,\vec{c}]=-\frac{2}{3}q$

$\therefore$ $\frac{5}{11}\times\sqrt{\frac{44}{125}}=-\frac{2}{3}q$

$\therefore$ $q=-\frac{3}{\sqrt{55}}$

17. $\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}^2=[\vec{a}\,\vec{b}\,\vec{c}]^2$

$=((\vec{a}\times\vec{b})\cdot\vec{c})^2$

$=(ab\sin\theta\,\vec{c}\cdot\vec{c})^2$

$=\frac{a^2b^2}{4}$

$=\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

Exercises

Single Correct Answer Type

1. (3) If $\vec{x} = \vec{y} \Rightarrow \hat{a} \cdot \vec{x} = \hat{a} \cdot \vec{y}$. This equality must hold for any arbitrary $\hat{a}$

2. (4) $\vec{a} \times (\vec{a} \times \vec{b}) = \vec{c} \Rightarrow |\vec{a}||\vec{a} \times \vec{b}| = |\vec{c}|$

$(\because \vec{a} \perp (\vec{a} \times \vec{b}))$

$1\,(1 \times 5)\sin\theta = 3 \Rightarrow \sin\theta = \frac{3}{5} \Rightarrow \tan\theta = \frac{3}{4}$.

3. (3) $|\vec{a} + \vec{b} + \vec{c}|^2 = 6$

or $|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$

$+ 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 6$

$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}|$ and

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cdot \cos\frac{\pi}{3}$

i.e., $\vec{a} \cdot \vec{b} = \frac{1}{2}|\vec{a}|^2$

$\therefore \quad 3|\vec{a}|^2 + 3|\vec{a}|^2 = 6$

$\Rightarrow |\vec{a}|^2$ or $|\vec{a}| = 1$

4. (2) Let $\vec{\alpha} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors, if $\vec{\alpha}$ makes angles θ, ϕ, ψ with $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, then

$$\vec{\alpha} \cdot \vec{a} = \frac{\vec{a} \cdot \vec{a}}{|\vec{a}|}$$

or $|\vec{\alpha}| \cdot |\vec{a}| \cos\theta = |\vec{a}|$

or $\cos\theta = \frac{1}{|\vec{\alpha}|}$

Similarly, $\cos\phi = \frac{1}{|\vec{\alpha}|}, \cos\psi = \frac{1}{|\vec{\alpha}|}$

$\therefore \quad \theta = \phi = \psi$

5. (3) $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ or $(\vec{r} - \vec{b}) \times \vec{a} = 0$

$\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ or $(\vec{r} - \vec{a}) \times \vec{b} = 0$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-2 & y & z+1 \\ 1 & 1 & 0 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-1 & y-1 & z \\ 2 & 0 & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad z + 1 = 0, x - y = 2$

and $\quad y - 1 = 0, x - 1 + 2z = 0$

$\Rightarrow \quad x = 3, y = 1, z = -1$

6. (4) $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$

$\Rightarrow \quad |\vec{a}||\vec{b}||\cos\theta| = |\vec{a}||\vec{b}||\sin\theta|$

(where θ is the angle between $\vec{a}$ and $\vec{b}$)

$\Rightarrow \quad |\cos\theta| = |\sin\theta|$

$\Rightarrow \quad \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ (as $0 \le \theta \le \pi$)

But $\vec{a} \cdot \vec{b} < 0$, therefore $\theta = \frac{3\pi}{4}$

7. (3) $|\vec{a} + \vec{b} + \vec{c}|^2 = 1$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{a}||\vec{b}|\cos\theta_1$

$+ 2|\vec{b}||\vec{c}|\cos\theta_2 + 2|\vec{c}||\vec{a}|\cos\theta_3 = 1$

$\Rightarrow \quad \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = -1$

Hence, one of θ_1, θ_2 and θ_3 should be an obtuse angle.

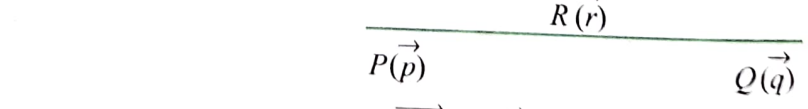
8. (2) $|\vec{a} \times \vec{b} - \vec{a} \times \vec{c}|^2$

$= |\vec{a} \times (\vec{b} - \vec{c})|^2$

$= |\vec{a}|^2 |\vec{b} - \vec{c}|^2 - (\vec{a} \cdot (\vec{b} - \vec{c}))^2 = |\vec{b} - \vec{c}|^2$

$= |\vec{b}|^2 + |\vec{c}|^2 - 2|\vec{b}||\vec{c}|\cos\frac{\pi}{3} = 1$

9. (3) $R(\vec{r})$ moves on PQ.



10. (2) $|\overrightarrow{AC} \times \overrightarrow{BD}|$

$= 2|\overrightarrow{AB} \times \overrightarrow{AD}|$

$$= 2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -5 \\ 1 & 2 & 3 \end{vmatrix}$$

$= |2[\hat{i}(12 + 10) - \hat{j}(6 + 5) + \hat{k}(4 - 4)]|$

$= |2[22\hat{i} - 11\hat{j}]|$

$= 22|[2\hat{i} - \hat{j}]|$

$= 22 \times \sqrt{5}$

11. (3) $(\hat{a} + \hat{b} + \hat{c})^2 \geq 0$

$\Rightarrow \quad 3 + 2(\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a}) \geq 0$

$\Rightarrow \quad 3 + 6\cos\theta \geq 0$

$\Rightarrow \quad \cos\theta \geq -\frac{1}{2}$

$\Rightarrow \quad \theta = \frac{2\pi}{3}$

12. (3) $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Similarly, $\vec{c} \times \vec{d}$ is a vector perpendicular to the plane containing $\vec{c}$ and $\vec{d}$.

Thus, the two planes will be parallel if their normals, i.e., $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$, are parallel. Thus,

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})) = \vec{0}$

13. (4) Let $\vec{r} \neq \vec{0}$. Then $\vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b} = \vec{r} \cdot \vec{c} = 0$

Hence, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, which is a contradiction.

Therefore, $\vec{r} = \vec{0}$

14. (3) $\vec{a} \times (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - \hat{k} = (\hat{j} \times (\hat{i} + 2\hat{j} + \hat{k}))$

or $\quad (\vec{a} - \hat{j}) \times (\hat{i} + 2\hat{j} + \hat{k}) = \vec{0}$

or $\quad \vec{a} - \hat{j} = \lambda(\hat{i} + 2\hat{j} + \hat{k})$

or $\quad \vec{a} = \lambda\hat{i} + (2\lambda + 1)\hat{j} + \lambda\hat{k}, \lambda \in R$

15. (1) $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + \vec{b}) = 0$

or $\quad 6|\vec{a}|^2 - 5|\vec{b}|^2 = 7\vec{a} \cdot \vec{b}$

Also, $(\vec{a} + 4\vec{b}) \cdot (\vec{b} - \vec{a}) = 0$

or $\quad -|\vec{a}|^2 + 4|\vec{b}|^2 = 3\vec{a} \cdot \vec{b}$

or $\quad \frac{6}{7}|\vec{a}|^2 - \frac{5}{7}|\vec{b}|^2 = -\frac{1}{3}|\vec{a}|^2 + \frac{4}{3}|\vec{b}|^2$

or $\quad 25|\vec{a}|^2 = 43|\vec{b}|^2$

$\Rightarrow \quad 3\vec{a} \cdot \vec{b} = -|\vec{a}|^2 + 4|\vec{b}|^2 = \frac{57}{25}|\vec{b}|^2$

or $\quad 3|\vec{a}||\vec{b}|\cos\theta = \frac{57}{25}|\vec{b}|^2$

or $\quad 3\sqrt{\frac{43}{25}}|\vec{b}|^2 \cos\theta = \frac{57}{25}|\vec{b}|^2$

or $\quad \cos\theta = \frac{19}{5\sqrt{43}}$

16. (1) Let l, m and n be the direction cosines of the required vector. Then, $l = m$ (given). Therefore

Required vector $\vec{r} = l\hat{i} + m\hat{j} + n\hat{k} = l\hat{i} + l\hat{j} + n\hat{k}$

Now, $l^2 + m^2 + n^2 = 1$ or $2l^2 + n^2 = 1$ (i)

Since, $\hat{r}$ is perpendicular to $-\hat{i} + 2\hat{j} + 2\hat{k}$,

$\vec{r} \cdot (-\hat{i} + 2\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \quad -l + 2l + 2n = 0$ or $l + 2n = 0$ (ii)

From (i) and (ii), we get: $n = \mp\frac{1}{3}$, $l = \pm\frac{2}{3}$

Hence, required vector

$\vec{r} = -(\pm 2\hat{i} \pm 2\hat{j} \quad \hat{k})$

$= \pm\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$

17. (4) The angle between $\vec{a}$ and $\vec{b}$ is obtuse. Therefore,

$\vec{a} \cdot \vec{b} < 0$

$\Rightarrow \quad 14x^2 - 8x + x < 0$

or $\quad 7x(2x - 1) < 0$

or $\quad 0 < x < 1/2$ (i)

The angle between $\vec{b}$ and the z-axis is acute and less than $\pi/6$. Therefore,

$\frac{\vec{b} \cdot \vec{k}}{|\vec{b}||\vec{k}|} > \cos\pi/6 \quad (\because \theta < \pi/6 \Rightarrow \cos\theta > \cos\pi/6)$

$\Rightarrow \quad \frac{x}{\sqrt{x^2 + 53}} > \frac{\sqrt{3}}{2}$

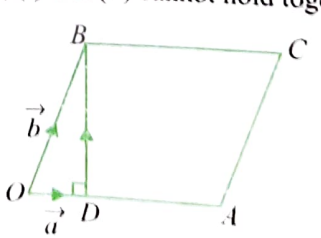
or $\quad 4x^2 > 3x^2 + 159$

or $\quad x^2 > 159$

$\Rightarrow \quad x > \sqrt{159}$ or $x < -\sqrt{159}$ (ii)

Clearly, (i) and (ii) cannot hold together.

18. (3)



Let $\overrightarrow{OD} = t\vec{a}$

$\therefore \quad \overrightarrow{DB} = \vec{b} - t\vec{a}$

or $\quad (\vec{b} - t\vec{a}) \cdot \vec{a} = 0 \qquad (\because \overrightarrow{DB} \perp \overrightarrow{OA})$

or $\quad t = \dfrac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}$

$\therefore \quad \overrightarrow{DB} = \vec{b} - \dfrac{(\vec{b} \cdot \vec{a})\vec{a}}{|\vec{a}|^2}$

19. (4) $(3\vec{a} + \vec{b}) \cdot (\vec{a} - 4\vec{b})$

$= 3|\vec{a}|^2 - 11\vec{a} \cdot \vec{b} - 4|\vec{b}|^2$

$= 3 \times 36 - 11 \times 6 \times 8 \cos\pi - 4 \times 64 > 0$

Therefore, the angle between $\vec{a}$ and $\vec{b}$ is acute.

The longer diagonal is given by

$\vec{\alpha} = (3\vec{a} + \vec{b}) + (\vec{a} - 4\vec{b}) = 4\vec{a} - 3\vec{b}$

Now, $|\vec{\alpha}|^2 = |4\vec{a} - 3\vec{b}|^2$

$= 16|\vec{a}|^2 + 9|\vec{b}|^2 - 24\vec{a} \cdot \vec{b}$

$= 16 \times 36 + 9 \times 64 - 24 \times 6 \times 8 \cos\pi$

$= 16 \times 144$

or $\quad |4\vec{a} - 3\vec{b}| = 48$

20. (2) $\vec{c} = m\vec{a} + n\vec{b} + p(\vec{a} \times \vec{b})$

Taking dot product with $\vec{a}$ and $\vec{b}$, we have

$m = n = \cos\theta$

$\Rightarrow \quad |\vec{c}| = |\cos\theta\,\vec{a} + \cos\theta\,\vec{b} + p(\vec{a} \times \vec{b})| = 1$

Squaring both sides, we get

$\cos^2\theta + \cos^2\theta + p^2 = 1$

or $\quad \cos\theta = \pm\dfrac{\sqrt{1 - p^2}}{\sqrt{2}}$

Now $\quad -\dfrac{1}{\sqrt{2}} \le \cos\theta \le \dfrac{1}{\sqrt{2}}$ (for real value of θ)

$\therefore \quad \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$

21. (1) $\vec{b} - 2\vec{c} = \vec{\ }$

or $\quad \vec{b} = 2\vec{c} + \lambda\vec{a}$

or $\quad |\vec{b}|^2 = |2\vec{c} + \lambda\vec{a}|^2$

$\Rightarrow \quad 16 = 4|\vec{c}|^2 + \lambda^2|\vec{a}|^2 + 4\lambda\,\vec{a} \cdot \vec{c}$

$= 4 + \lambda^2 + 4\lambda\dfrac{1}{4}$

or $\quad \lambda^2 + \lambda - 12 = 0$

or $\quad \lambda = 3, -4$

22. (1) A vector perpendicular to the plane of O, P and Q is $\overrightarrow{OP} \times \overrightarrow{OQ}$.

Now, $\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & \lambda \\ 2 & -1 & \lambda \end{vmatrix}$

$= 2\lambda\hat{i} - 2\lambda\hat{j} - 6\hat{k}$

Therefore, $\hat{i} - \hat{j} + 6\hat{k}$ is parallel to

$2\lambda\hat{i} - 2\lambda\hat{j} - 6\hat{k}$

Hence, $\dfrac{1}{2\lambda} = \dfrac{-1}{-2\lambda} = \dfrac{6}{-6}$

$\lambda = -\dfrac{1}{2}$

23. (1) A vector coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$ is $\lambda\left((\vec{a} \times \vec{b}) \times \vec{c}\right)$.

But $\quad \lambda\left((\vec{a} \times \vec{b}) \times \vec{c}\right) = \lambda\left[(\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}\right]$

$= \lambda\left[4\vec{b} - 4\vec{a}\right]$

$= 4\lambda\left[\hat{j} - \hat{k}\right]$

Now $4|\lambda|\sqrt{2} = \sqrt{2}$ (Given) $\Rightarrow \lambda = \pm\dfrac{1}{4}$

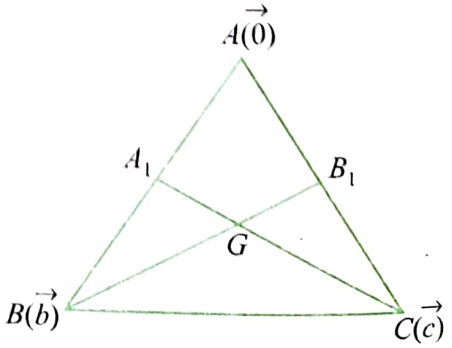
Hence, the required vector is $\hat{j} - \hat{k}$ or $-\hat{j} + \hat{k}$

24. (1) $\vec{a} - \vec{p} + \vec{b} - \vec{p} + \vec{c} - \vec{p} = 0$

or $\quad \vec{p} = \dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$

Hence, P is centroid.

25. (2)



Let P.V. of A, B and C be $\vec{0}, \vec{b}$ and $\vec{c}$, respectively. Therefore,

$$\vec{G} = \frac{\vec{b}+\vec{c}}{3}$$

$$\vec{A_1} = \frac{\vec{b}}{2}, \vec{B_1} = \frac{\vec{c}}{2}$$

$$\Delta_{AB_1G} = \frac{1}{2}|\overrightarrow{AG} \times \overrightarrow{AB_1}|$$

$$= \frac{1}{2}\left|\frac{\vec{b}+\vec{c}}{3} \times \left(\frac{\vec{c}}{2}\right)\right|$$

$$= \frac{1}{12}|\vec{b} \times \vec{c}|$$

$$\Delta_{AA_1G} = \frac{1}{2}|\overrightarrow{AG} \times \overrightarrow{AA_1}| = \frac{1}{2}\left|\frac{\vec{b}+\vec{c}}{3} \times \left(\frac{\vec{b}}{2}\right)\right|$$

$$= \frac{1}{12}|\vec{b} \times \vec{c}|$$

$$\Rightarrow \quad \Delta_{GA_1AB_1} = \frac{1}{6}|\vec{b} \times \vec{c}| = \frac{1}{3}\cdot\frac{1}{2}|\vec{b} \times \vec{c}|$$

$$= \frac{1}{3}\Delta_{ABC}$$

$$\Rightarrow \quad \frac{\Delta}{\Delta_1} = 3$$

26. (1) Points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar. Therefore,

$$\sin\alpha + 2\sin 2\beta + 3\sin 3\gamma = 1$$

Now $|\sin\alpha + 2\sin 2\beta + 3\sin 3\gamma|$

$$\le \sqrt{1+4+9}\cdot\sqrt{\sin^2\alpha + \sin^2 2\beta + \sin^2 3\gamma}$$

or $\quad \sin^2\alpha + \sin^2 2\beta + \sin^2 3\gamma \ge \frac{1}{14}$

27. (3) $1 + 9(\vec{a}\cdot\vec{b})^2 - 6(\vec{a}\cdot\vec{b}) + 4|\vec{a}|^2$

$$+|\vec{b}|^2 + 9|\vec{a} \times \vec{b}|^2 + 4\vec{a}\cdot\vec{b} = 47$$

or $\quad 1 + 4 + 4 + 36 - 4\cos\theta = 47$

or $\quad \cos\theta = -\frac{1}{2}$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$.

28. (3) $k = |2(\vec{a} \times \vec{b})| + |3(\vec{a}\cdot\vec{b})|$

$$= 12\sin\theta + 18\cos\theta$$

$\Rightarrow$ Maximum value of $k = \sqrt{12^2+18^2} = 6\sqrt{13}$

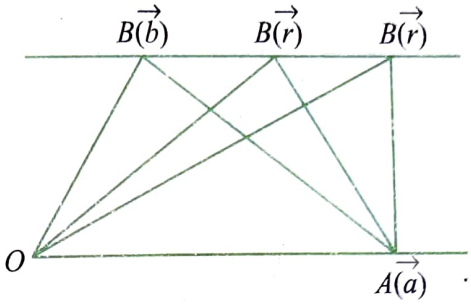
29. (2) $|\vec{a} + \vec{b} + 3\vec{c}|^2 = 16$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 9|\vec{c}|^2 + 2\cos\theta_1 + 6\cos\theta_2 + 6\cos\theta_3 = 16, \ \theta_3 \in [\pi/6, 2\pi/3]$$

or $\quad 2\cos\theta_1 + 6\cos\theta_2 = 5 - 6\cos\theta_3$

or $\quad (\cos\theta_1 + 3\cos\theta_2)_{max} = 4$

30. (3) $|\vec{a} \times \vec{r}| = |\vec{c}|$



Triangles on the same base and between the same parallel will have the same area.

31. (3) Given $\vec{v}\cdot\vec{u} = \vec{w}\cdot\vec{u}$

and $\quad \vec{v} \perp \vec{w} \Rightarrow \vec{v}\cdot\vec{w} = 0$

Now, $|\vec{u} - \vec{v} + \vec{w}|^2$

$$= |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{u}\cdot\vec{v} - 2\vec{w}\cdot\vec{v} + 2\vec{u}\cdot\vec{w}$$

$$= 1 + 4 + 9$$

so $\quad |\vec{u} - \vec{v} + \vec{w}| = \sqrt{14}$

32. (2) We have

$$\vec{p}\cdot\vec{q} = 0$$

$$\Rightarrow \quad (5\vec{a} - 3\vec{b})\cdot(-\vec{a} - 2\vec{b}) = 0$$

$$\Rightarrow \quad 6|\vec{b}|^2 - 5|\vec{a}|^2 - 7\vec{a}\cdot\vec{b} = 0 \qquad \text{(i)}$$

Also $\quad \vec{r}\cdot\vec{s} = 0$

$$\Rightarrow \quad (-4\vec{a} - \vec{b})\ (-\vec{a} + \vec{b}) = 0$$

$$\Rightarrow \quad 4|\vec{a}|^2 - |\vec{b}|^2 - 3\vec{a}\cdot\vec{b} = 0 \qquad \text{(ii)}$$

Now $\quad \vec{x} = \frac{1}{3}(\vec{p} + \vec{r} + \vec{s})$

$$= \frac{1}{3}(5\vec{a} - 3\vec{b} - 4\vec{a} - \vec{b} - \vec{a} + \vec{b}) = -\vec{b}$$

and $\quad \vec{y} = \frac{1}{5}(\vec{r} + \vec{s}) = \frac{1}{5}(-5\vec{a}) = -\vec{a}$

Angle between $\vec{x}$ and $\vec{y}$, i.e.,

$$\cos\theta = \frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \quad \text{(iii)}$$

From (i) and (ii),

$$|\vec{a}| = \sqrt{\frac{25}{19}}\sqrt{\vec{a}\cdot\vec{b}}$$

and $$|\vec{b}| = \sqrt{\frac{43}{19}}\sqrt{\vec{a}\cdot\vec{b}}$$

$$\therefore \quad |\vec{a}||\vec{b}| = \frac{\sqrt{25\times 43}}{19}\vec{a}\cdot\vec{b}$$

$$\theta = \cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$$

33. (1) $\vec{\alpha}\parallel(\vec{\beta}\times\vec{\gamma}) \Rightarrow \vec{\alpha}\perp\vec{\beta}$ and $\vec{\alpha}\perp\vec{\gamma}$

Now, $(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\gamma})$

$$= |\vec{\alpha}|^2(\vec{\beta}\cdot\vec{\gamma}) - (\vec{\alpha}\cdot\vec{\beta})(\vec{\alpha}.\vec{\gamma})$$

$$= |\vec{\alpha}|^2\cdot(\vec{\beta}\cdot\vec{\gamma})$$

34. (2) Since $\overrightarrow{OA} = \hat{i}+\hat{j}+\hat{k}$

$$\overrightarrow{OB} = \hat{i}+5\hat{j}-\hat{k}$$

$$\overrightarrow{OC} = 2\hat{i}+3\hat{j}+5\hat{k}$$

$$a = BC = |\overrightarrow{BC}| = |\overrightarrow{OC} - \overrightarrow{OB}|$$

$$= |\hat{i}-2\hat{j}+6\hat{k}| = \sqrt{41}$$

$$b = CA = |\overrightarrow{CA}| = |\overrightarrow{OA} - \overrightarrow{OC}|$$

$$= |-\hat{i}-2\hat{j}-4\hat{k}| = \sqrt{21}$$

and $$c = AB = |\overrightarrow{AB}| = |\overrightarrow{OB} - \overrightarrow{OA}|$$

$$= |0\hat{i}+4\hat{j}-2\hat{k}| = \sqrt{20}$$

Since $a > b > c$, A is the greatest angle. Therefore,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{21+20-41}{2\cdot\sqrt{21}\cdot\sqrt{20}} = 0$$

$$\therefore \quad \angle A = 90^\circ$$

35. (2) $\vec{a}+\vec{b} = \lambda\vec{c}$ (i)

and $\vec{b}+\vec{c} = \mu\vec{a}$ (ii)

$\therefore$ $(\lambda\vec{c}-\vec{a})+\vec{c} = \mu\vec{a}$ (putting $\vec{b} = \lambda\vec{c}-\vec{a}$)

$$\Rightarrow \quad (\lambda+1)\vec{c} = (\mu+1)\vec{a}$$

$$\Rightarrow \quad \lambda = \mu = -1$$

$$\Rightarrow \quad \vec{a}+\vec{b}+\vec{c} = 0$$

or $$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2 + 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

or $$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -3$$

36. (4) $\vec{0} = (\vec{a}+\vec{b})\cdot(2\vec{a}+3\vec{b})\times(3\vec{a}-2\vec{b})$

$$= (\vec{a}+\vec{b})\cdot(-4\vec{a}\times\vec{b}-9\vec{a}\times\vec{b})$$

$$= -13\,(\vec{a}+\vec{b})\cdot(\vec{a}\times\vec{b})$$

which is true for all values of $\vec{a}$ and $\vec{b}$.

37. (3) We have

$$\overrightarrow{AB}\cdot\overrightarrow{AC} + \overrightarrow{BC}\cdot\overrightarrow{BA} + \overrightarrow{CA}\cdot\overrightarrow{CB}$$

$$= (AB)(AC)\cos\theta + (BC)(BA)\sin\theta + 0$$

$$= AB\,(AC\cos\theta + BC\sin\theta)$$

$$= AB\left(\frac{(AC)^2}{AB} + \frac{(BC)^2}{AB}\right)$$

$$= AC^2 + BC^2 = AB^2 = p^2$$

38. (3) $\vec{a}_1 = (\vec{a}\cdot\hat{b})\hat{b} = \dfrac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$

$$\Rightarrow \vec{a}_2 = \vec{a} - \vec{a}_1 = \vec{a} - \frac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$$

Thus, $$\vec{a}_1\times\vec{a}_2 = \frac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2} \times \left(\vec{a} - \frac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right)$$

$$= \frac{(\vec{a}\cdot\vec{b})(\vec{b}\times\vec{a})}{|\vec{b}|^2}$$

39. (2) Let the required vector be $\vec{r}$. Then

$$\vec{r} = x_1\vec{b} + x_2\vec{c} \text{ and } \vec{r}\cdot\vec{a} = \sqrt{\frac{2}{3}}\,(|\vec{a}|) = 2$$

Now, $\vec{r}\cdot\vec{a} = x_1\vec{a}\cdot\vec{b} + x_2\vec{a}\cdot\vec{c}$

$$\Rightarrow \quad 2 = x_1(2-2-1) + x_2(2-1-2)$$

$$\Rightarrow \quad x_1 + x_2 = -2$$

$$\Rightarrow \quad \vec{r} = x_1(\hat{i}+2\hat{j}-\hat{k}) + x_2(\hat{i}+\hat{j}-2\hat{k})$$

$$= \hat{i}(x_1 + x_2) + \hat{j}(2x_1 + x_2) - \hat{k}(2x_2 + x_1)$$

$$= -2\hat{i} + \hat{j}(x_1 - 2) - \hat{k}(-4 - x_1),$$

where $x_1 \in R$

40. (1) Let P.V. of P, A, B and C be $\vec{p}$, $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively, and $O(\vec{0})$ be the circumcentre of equilateral triangle ABC. Then

$$|\vec{p}| = |\vec{b}| = |\vec{a}| = |\vec{c}| = \frac{l}{\sqrt{3}}$$

Now $|\overrightarrow{PA}|^2 = |\vec{a} - \vec{p}|^2 = |\vec{a}|^2 + |\vec{p}|^2 - 2\vec{p}\cdot\vec{a}$

Similarly, $|\overrightarrow{PB}|^2 = |\vec{b}|^2 + |\vec{p}|^2 - 2\vec{p}\cdot\vec{b}$

and $|\vec{PC}|^2 = |\vec{c}|^2 + |\vec{p}|^2 - 2\vec{p}\cdot\vec{c}$

$$\Rightarrow \quad \Sigma|\overrightarrow{PA}|^2 = 6\cdot\frac{l^2}{3} - 2\vec{p}\cdot(\vec{a} + \vec{b} + \vec{c})$$

$$= 2l^2 \quad (\text{as } \vec{a} + \vec{b} + \vec{c}/3 = \vec{0})$$

41. (4) For minimum value $|\vec{r} + b\vec{s}| = 0$.

Let $\vec{r}$ and $\vec{s}$ are anti-parallel so $b\vec{s} = -\vec{r}$

$$\therefore \quad |b\vec{s}|^2 + |\vec{r} + b\vec{s}|^2 = |-\vec{r}|^2 + |\vec{r} - \vec{r}|^2 = |\vec{r}|^2$$

42. (3) Let the required vector $\vec{r}$ be such that

$$\vec{r} = x_1\vec{a} + x_2\vec{b} + x_3\vec{a}\times\vec{b}$$

We must have $\vec{r}\cdot\vec{a} = \vec{r}\cdot\vec{b} = \vec{r}\cdot(\vec{a}\times\vec{b})$ (as $\vec{r}, \vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are unit vectors and $\vec{r}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$)

Now $\vec{r}\cdot\vec{a} = x_1$, $\vec{r}\cdot\vec{b} = x_2$, $\vec{r}\cdot(\vec{a}\times\vec{b}) = x_3$

$$\Rightarrow \quad \vec{r} = \lambda(\vec{a} + \vec{b} + (\vec{a}\times\vec{b}))$$

Also, $\vec{r}\cdot\vec{r} = 1$

$$\Rightarrow \quad \lambda^2(\vec{a} + \vec{b} + \vec{a}\times\vec{b})\cdot(\vec{a} + \vec{b} + (\vec{a}\times\vec{b})) = 1$$

or $\quad \lambda^2(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{a}\times\vec{b}|^2) = 1$

or $\quad \lambda^2 = \frac{1}{3}$

or $\quad \lambda = \pm\frac{1}{\sqrt{3}}$

$$\Rightarrow \quad \vec{r} = \pm\frac{1}{\sqrt{3}}(\vec{a} + \vec{b} + \vec{a}\times\vec{b})$$

43. (4) $\vec{a} + \vec{b} = \mu\vec{p} \qquad \vec{b}\cdot\vec{q} = 0\,,\ |\vec{b}|^2 = 1$

$$\because \quad \vec{a} + \vec{b} = \mu\vec{p}$$

$$\Rightarrow \quad (\vec{a} + \vec{b})\times\vec{a} = \mu\vec{p}\times\vec{a}\,,\ \vec{b}\times\vec{a} = \mu\vec{p}\times\vec{a}$$

$$\Rightarrow \quad \vec{q}\times(\vec{b}\times\vec{a}) = \mu\vec{q}\times(\vec{p}\times\vec{a})$$

$$\Rightarrow \quad (\vec{q}\cdot\vec{a})\vec{b} - (\vec{q}\cdot\vec{b})\vec{a} = \mu\vec{q}\times(\vec{p}\times\vec{a})$$

$$\Rightarrow \quad (\vec{q}\cdot\vec{a})\vec{b} = \mu\vec{q}\times(\vec{p}\times\vec{a})$$

$$\because \quad \vec{a} + \vec{b} = \mu\vec{p}$$

$$\Rightarrow \quad \vec{q}\cdot(\vec{a} + \vec{b}) = \mu\vec{q}\cdot\vec{p}$$

$$\Rightarrow \quad \vec{q}\cdot\vec{a} + \vec{q}\cdot\vec{b} = \mu\vec{p}\cdot\vec{q}$$

$$\Rightarrow \quad \mu = \frac{\vec{q}\cdot\vec{a}}{\vec{p}\cdot\vec{q}}$$

$$\Rightarrow \quad (\vec{q}\cdot\vec{a})\vec{b} = \frac{\vec{q}\cdot\vec{a}}{\vec{p}\cdot\vec{q}}[(\vec{q}\cdot\vec{a})\cdot\vec{p} - (\vec{q}\cdot\vec{p})\vec{a}]$$

$$\Rightarrow \quad |(\vec{q}\cdot\vec{a})\vec{p} - (\vec{q}\cdot\vec{p})\vec{a}|$$

$$= |(\vec{p}\cdot\vec{q})\vec{b}| = |(\vec{p}\cdot\vec{q})|\cdot|\vec{b}|$$

$$\Rightarrow \quad |(\vec{q}\cdot\vec{a})\vec{p} - (\vec{q}\cdot\vec{p})\vec{a}| = |\vec{p}.\vec{q}|$$

44. (3) $\vec{d}\cdot\hat{a} = \vec{d}\cdot\hat{b} = \vec{d}\cdot\hat{c}$

$$\Rightarrow \quad \lambda(\hat{a}\cdot\hat{b} + \hat{a}\cdot\hat{c}) = \lambda(1 + \hat{b}\cdot\hat{c}) = \lambda(1 + \hat{b}\cdot\hat{c})$$

or $\quad 1 + \hat{b}\cdot\hat{c} = \hat{a}\cdot\hat{b} + \hat{a}\cdot\hat{c}$

or $\quad 1 - \hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} - \hat{a}\cdot\hat{c} = 0$

or $\quad 1 - \hat{a}\cdot\hat{b} + (\hat{b} - \hat{a})\cdot\hat{c} = 0$

or $\quad \hat{a}\cdot(\hat{a} - \hat{b}) + (\hat{b} - \hat{a})\cdot\hat{c} = 0$

or $\quad (\hat{a} - \hat{c})\cdot(\hat{a} - \hat{b}) = 0$ Hence $\hat{a} - \hat{c}$ is perpendicular to $(\hat{a} - \hat{b})$ i.e., The triangle is right angled.

45. (3) The given relation can be rewritten as the vector expression

$$(\sqrt{a^2 - 4}\,\hat{i} + a\hat{j} + \sqrt{a^2 + 4}\,\hat{k})$$

$$\cdot(\tan A\,\hat{i} + \tan B\,\hat{j} + \tan C\,\hat{k}) = 6a$$

$$\Rightarrow \quad \left(\sqrt{a^2 - 4 + a^2 + a^2 + 4}\right)$$

$$\left(\sqrt{\tan^2 A + \tan^2 B + \tan^2 C}\right)\cdot(\cos\theta) = 6a$$

$$(\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta)$$

$$\sqrt{3}\, a \sqrt{\tan^2 A + \tan^2 B + \tan^2 C} \cdot (\cos\theta) = 6a$$

$$\tan^2 A + \tan^2 B + \tan^2 C = 12 \sec^2\theta \geq 12$$

$(\because \sec^2\theta \geq 1)$

The least value of $\tan^2 A + \tan^2 B + \tan^2 C$ is 12.

46. (4) $\Delta = \frac{1}{2}|(\hat{j} + \lambda\hat{k}) \times (\hat{i} + \lambda\hat{k})| = \frac{1}{2}|-\hat{k} + \lambda\hat{i} + \lambda\hat{j}|$

$$= \frac{1}{2}\sqrt{2\lambda^2 + 1}$$

$$\Rightarrow \quad \frac{9}{4} \leq \frac{1}{4}(2\lambda^2 + 1) \leq \frac{33}{4}$$

or $\quad 4 \leq \lambda^2 \leq 16$

or $\quad 2 \leq |\lambda| \leq 4$

47. (3) Let the projection be x, then

$$\vec{a} = \frac{x(\hat{i} + \hat{j})}{\sqrt{2}} + \frac{x(-\hat{i} + \hat{j})}{\sqrt{2}} + x\,\hat{k}$$

$$\therefore \quad \vec{a} = \frac{2x\,\hat{j}}{\sqrt{2}} + x\,\hat{k} \quad \text{or} \quad \hat{a} = \frac{\sqrt{2}}{\sqrt{3}}\hat{j} + \frac{\hat{k}}{\sqrt{3}}$$

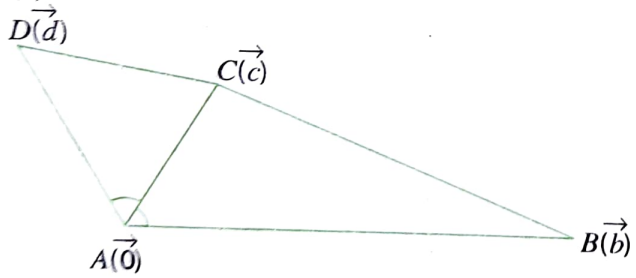
48. (2) Let $\vec{r}$ be the new position. Then $\vec{r} = \lambda\hat{k} + \mu(\hat{i} + \hat{j})$

Also $\quad \vec{r} \cdot \hat{k} = -\frac{1}{\sqrt{2}} \Rightarrow \lambda = -\frac{1}{\sqrt{2}}$

Also, $\lambda^2 + 2\mu^2 = 1 \Rightarrow 2\mu^2 = \frac{1}{2}$ or $\mu = \pm\frac{1}{2}$

$$\therefore \quad \vec{r} = \pm\frac{1}{2}(\hat{i} + \hat{j}) - \frac{\hat{k}}{\sqrt{2}}$$

49. (3)



Let $|\overrightarrow{AC}| = \lambda > 0$

Then from $15|\overrightarrow{AC}| = 3|\overrightarrow{AB}| = 5|\overrightarrow{AD}|$

$|\overrightarrow{AB}| = 5\lambda$

Let θ be the angle between $\overrightarrow{BA}$ and $\overrightarrow{CD}$. Then

$$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{CD}}{|\overrightarrow{BA}||\overrightarrow{CD}|} = \frac{-\vec{b} \cdot (\vec{d} - \vec{c})}{|\vec{b}||\vec{d} - \vec{c}|} \quad \text{(i)}$$

Now $\quad -\vec{b} \cdot (\vec{d} - \vec{c}) = \vec{b} \cdot \vec{c} - \vec{b} \cdot \vec{d}$

$$= |\vec{b}||\vec{c}| \cos\frac{\pi}{3} - |\vec{b}||\vec{d}| \cos\frac{2\pi}{3}$$

$$= (5\lambda)(\lambda)\frac{1}{2} + (5\lambda)(3\lambda)\frac{1}{2}$$

$$= \frac{5\lambda^2 + 15\lambda^2}{2}$$

$$= 10\lambda^2$$

Denominator of (i) $= |\vec{b}||\vec{d} - \vec{c}|$

Now $|\vec{d} - \vec{c}|^2 = |\vec{d}|^2 + |\vec{c}|^2 - 2\vec{c} \cdot \vec{d}$

$$= 9\lambda^2 + \lambda^2 - 2(\lambda)(3\lambda)(1/2)$$
$$= 10\lambda^2 - 3\lambda^2$$
$$= 7\lambda^2$$

Denominator of (i) $= (5\lambda)(\sqrt{7}\,\lambda) = 5\sqrt{7}\,\lambda^2$

$$\therefore \quad \cos\theta = \frac{10\lambda^2}{5\sqrt{7}\,\lambda^2} = \frac{2}{\sqrt{7}}$$

50. (1) Let A be the origin. $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$

So, $\quad \overrightarrow{AE} = \vec{b} + \frac{3}{2}\vec{a}$, $\overrightarrow{AG} = \vec{a} + 3\vec{b}$.

So, Required ratio

$$= \frac{\frac{1}{2}\left|(\vec{a} + 3\vec{b}) \times \left(\vec{b} + \frac{3}{2}\vec{a}\right)\right|}{\frac{1}{2}|\vec{a} \times \vec{b}|}$$

$$= \frac{7}{2}$$

51. (2) Let $\vec{a} = \lambda\vec{b} + \mu\vec{c}$

$\vec{a}$ is equally inclined to $\vec{b}$ and $\vec{d}$, where $\vec{d} = \hat{j} + 2\hat{k}$.

$$\Rightarrow \quad \frac{\vec{a} \cdot \vec{b}}{ab} = \frac{\vec{a} \cdot \vec{b}}{ab}$$

$$\Rightarrow \quad \frac{(\lambda\vec{b} + \mu\vec{c}) \cdot \vec{b}}{b} = \frac{(\lambda\vec{b} + \mu\vec{c}) \cdot \vec{d}}{d}$$

$$\frac{[\lambda(2\hat{i} + \hat{j}) + \mu(\hat{i} - \hat{j} + \hat{k})] \cdot (2\hat{i} + \hat{j})}{\sqrt{5}}$$

$$= \frac{[\lambda(2\hat{i} + \hat{j}) + \mu(\hat{i} - \hat{j} + \hat{k})] \cdot (\hat{j} + 2\hat{k})}{\sqrt{5}}$$

or $\lambda(4+1)+\mu(2-1)=\lambda(1)+\mu(-1+2)$

or $4\lambda=0$, i.e., $\lambda=0$

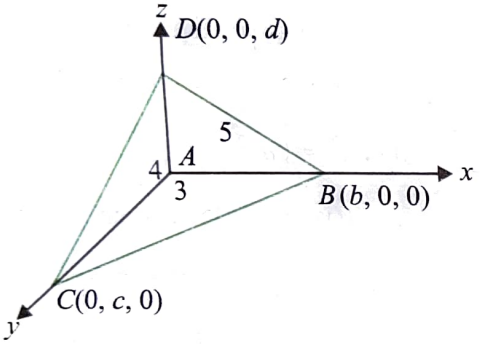
$\therefore \quad \hat{a}=\dfrac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$

52. (1) Area of $\Delta BCD=\dfrac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BD}|$

$=\dfrac{1}{2}|(b\hat{i}-c\hat{j})\times(b\hat{i}-d\hat{k})|$

$=\dfrac{1}{2}|bd\,\hat{j}+bc\,\hat{k}+dc\,\hat{i}|$

$=\dfrac{1}{2}\sqrt{b^2c^2+c^2d^2+d^2b^2}$ (i)



Now, $6=bc$; $8=cd$; $10=bd$

$b^2c^2+c^2d^2+d^2b^2=200$

Substituting the value in (i), we get

$A=\dfrac{1}{2}\sqrt{200}=5\sqrt{2}$

53. (4) $\vec{f}\left(\dfrac{5}{4}\right)=\left[\dfrac{5}{4}\right]\hat{i}+\left(\dfrac{5}{4}-\left[\dfrac{5}{4}\right]\right)\hat{j}+\left[\dfrac{5}{4}+1\right]\hat{k}$

$=\hat{i}+\left(\dfrac{5}{4}-1\right)\hat{j}+2\hat{k}$

$=\hat{i}+\dfrac{1}{4}\hat{j}+2\hat{k}$

When $0<t<1$, $\vec{f}(t)=0\,\vec{i}+\{t-0\}\,\vec{j}+\vec{k}$

$=t\,\vec{j}+\vec{k}$

$\vec{f}\left(\dfrac{5}{4}\right)\cdot\vec{f}(t)=2+\dfrac{t}{4}$

So $\cos\theta=\dfrac{2+\frac{t}{4}}{\left|\vec{i}+\frac{1}{4}\vec{j}+2\vec{k}\right||t\,\vec{j}+\vec{k}|}$

$=\dfrac{2+\frac{t}{4}}{\sqrt{1+\frac{1}{16}+4}\sqrt{1+t^2}}=\dfrac{8+t}{9\sqrt{1+t^2}}$

54. (1) $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{u}$, where $\vec{u}=\vec{a}\times\vec{c}$

$\Rightarrow \quad \vec{a}\cdot(\vec{b}\times\vec{u})=\vec{a}\cdot[\vec{b}\times(\vec{a}\times\vec{c})]$

$=\vec{a}\cdot[(\vec{b}\cdot\vec{c})\,\vec{a}-(\vec{a}\cdot\vec{b})\,\vec{c}]$

$=\vec{a}\cdot(\vec{b}\cdot\vec{c})\,\vec{a}\quad(\because\ \vec{a}\cdot\vec{b}=0)$

$=|\vec{a}|^2(\vec{b}\cdot\vec{c})$

55. (4) $(\hat{i}+\hat{j})\times(\hat{j}+\hat{k})=\hat{i}-\hat{j}+\hat{k}$ so that unit vector perpendicular to the plane of $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$ is

$\dfrac{1}{\sqrt{3}}(\hat{i}-\hat{j}+\hat{k})$.

Similarly, the other two unit vectors are

$\dfrac{1}{\sqrt{3}}(\hat{i}+\hat{j}-\hat{k})$ and $\dfrac{1}{\sqrt{3}}(-\hat{i}+\hat{j}+\hat{k})$.

Required volume $=\dfrac{3}{\sqrt{3}}\begin{vmatrix}1&-1&1\\1&1&-1\\-1&1&1\end{vmatrix}=4\sqrt{3}$

56. (3) $\vec{d}\cdot\vec{c}=\vec{d}\cdot\vec{b}=\vec{d}\cdot\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]$

Then $|(\vec{d}\cdot\vec{c})(\vec{a}\times\vec{b})+(\vec{d}\cdot\vec{a})$

$(\vec{b}\times\vec{c})+(\vec{d}\cdot\vec{b})(\vec{c}\times\vec{a})|=0$

or $[\vec{a}\,\vec{b}\,\vec{c}]|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|=0$

or $[\vec{a}\,\vec{b}\,\vec{c}]=0$ ($\because\ \vec{d}$ is non-zero)

Hence, $\vec{a},\vec{b},\vec{c}$ are coplanar.

57. (1) $(\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times\vec{b}))))$

$=(\vec{a}\times(\vec{a}\times((\vec{a}\cdot\vec{b})\,\vec{a}-(\vec{a}\cdot\vec{b})\,\vec{b})))$

$=(\vec{a}\times(\vec{a}\times(-4\vec{b})))$

$=-4(\vec{a}\times(\vec{a}\times\vec{b}))$

$=-4((\vec{a}\cdot\vec{b})\,\vec{a}-(\vec{a}\cdot\vec{a})\,\vec{b})$

$=-4(-4\vec{b})=16\,\vec{b}=48\,\hat{b}$

58. (4) Let $\vec{a}=6\hat{i}+6\hat{k}$, $\vec{b}=4\hat{j}+2\hat{k}$, $\vec{c}=4\hat{j}-8\hat{k}$

Then $\vec{a}\times\vec{b}=-24\hat{i}-12\hat{j}+24\hat{k}$

$=12(-2\hat{i}-\hat{j}+2\hat{k})$

$\therefore$ Area of the base of the parallelepiped

$$= \frac{1}{2}|\vec{a}\times\vec{b}|$$

$$= \frac{1}{2}(12\times 3)$$

$$= 18$$

Height of the parallelepiped

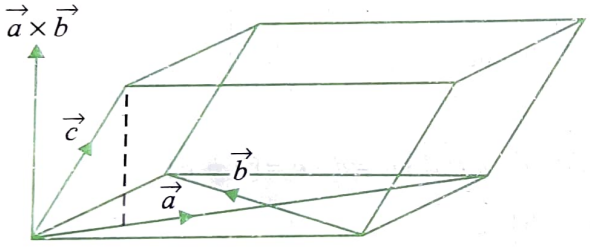
= Length of projection of $\vec{c}$ on $\vec{a}\times\vec{b}$

$$= \frac{|\vec{c}\cdot\vec{a}\times\vec{b}|}{|\vec{a}\times\vec{b}|}$$

$$= \frac{|12(-4-16)|}{36}$$

$$= \frac{20}{3}$$

$\therefore$ Volume of the parallelepiped $= 18\times\frac{20}{3} = 120$



59. (3) $3 = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

or $[\vec{a}\ \vec{b}\ \vec{c}] = 18$

Volume of the required parallelepiped

$$= [\vec{a}+\vec{b}\ \vec{b}+\vec{c}\ \vec{c}+\vec{a}]$$

$$= 2[\vec{a}\ \vec{b}\ \vec{c}] = 36$$

60. (2) Here $[\vec{a}\ \vec{b}\ \vec{c}] = \pm 1$

$$[\vec{a}+\vec{b}+\vec{c}\ \vec{a}+\vec{b}\ \vec{b}+\vec{c}]$$

$$= (\vec{a}+\vec{b}+\vec{c})\times(\vec{a}+\vec{b})\cdot(\vec{b}+\vec{c})$$

$$= \vec{c}\times(\vec{a}+\vec{b})\cdot(\vec{b}+\vec{c})$$

$$= (\vec{c}\times\vec{a}+\vec{c}\times\vec{b})\cdot(\vec{b}+\vec{c})$$

$$= \vec{c}\times\vec{a}\cdot\vec{b} = [\vec{a}\ \vec{b}\ \vec{c}] = \pm 1$$

61. (1) Let $\vec{c} = \lambda(\vec{a}\times\vec{b})$.

Hence, $\lambda(\vec{a}\times\vec{b})\cdot(\hat{i}+2\hat{j}-7\hat{k}) = 10$

$$\Rightarrow \lambda\begin{vmatrix} 2 & -3 & 1 \\ 1 & -2 & 3 \\ 1 & 2 & -7 \end{vmatrix} = 10$$

or $\lambda = -1$

$$\Rightarrow \vec{c} = -(\vec{a}\times\vec{b})$$

62. (4) $\vec{a}\perp\vec{b} \Rightarrow x-y+2=0$

$\vec{a}\cdot\vec{c} = 4 \Rightarrow x+2y=4$

Solving we get $x=0;\ y=2$

$$\Rightarrow \vec{a} = 2\hat{j}+2\hat{k}$$

$$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 0 & 2 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 8 = |\vec{a}|^2$$

63. (3) $(\vec{a}\times\vec{b}\cdot\vec{c})^2 = |\vec{a}|^2|\vec{b}|^2|\vec{c}|^2\sin^2\theta\cos^2\phi$

(θ is the angle between $\vec{a}$ and $\vec{b}$, $\phi = 0$)

$$= \frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$$

64. (3) $\vec{r}\cdot\vec{a} = 0, |\vec{r}\times\vec{b}| = |\vec{r}||\vec{b}|$ and $|\vec{r}\times\vec{c}| = |\vec{r}||\vec{c}|$

$$\Rightarrow \vec{r}\perp\vec{a},\vec{b},\vec{c}$$

$$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = 0$$

65. (2) $\vec{c} = \lambda(\vec{a}\times\vec{b})$

or $\vec{c}\cdot\vec{c} = \lambda(\vec{a}\times\vec{b})\cdot\vec{c}$

$$\Rightarrow \frac{1}{3} = \lambda$$

Also $|\vec{c}|^2 = \lambda^2|\vec{a}\times\vec{b}|^2$

$$\Rightarrow \frac{1}{3} = \frac{1}{9}(a^2b^2\sin^2\theta) = \frac{1}{9}\times 2\times 3\sin^2\theta$$

or $\sin^2\theta = \frac{1}{2}$

or $\theta = \frac{\pi}{4}$

66. (3) $4\vec{a}+5\vec{b}+9\vec{c} = 0$

$\Rightarrow$ Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

$\Rightarrow$ $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are collinear

$\Rightarrow$ $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = \vec{0}$.

(1) $[\vec{a}\times\vec{b}\ \ \vec{a}\times\vec{c}\ \ \vec{d}]$

$= (\vec{a}\times\vec{b})\cdot((\vec{a}\times\vec{c})\times\vec{d})$

$= (\vec{a}\times\vec{b})\cdot((\vec{a}\cdot\vec{d})\vec{c} - (\vec{c}\cdot\vec{d})\vec{a})$

$= (\vec{a}\cdot\vec{d})[\vec{a}\ \vec{b}\ \vec{c}]$

(1) Let $\vec{r} = x_1\hat{a} + x_2\hat{b} + x_3(\hat{a}\times\hat{b})$

$\Rightarrow \quad \vec{r}\cdot\hat{a} = x_1 + x_2\hat{a}\cdot\hat{b} + x_3\hat{a}\cdot(\hat{a}\times\hat{b}) = x_1$

Also, $\vec{r}\cdot\hat{b} = x_1\hat{a}\cdot\hat{b} + x_2 + x_3\hat{b}\cdot(\hat{a}+\hat{b}) = x_2$

and $\vec{r}\cdot(\hat{a}\times\hat{b}) = x_1\hat{a}\cdot(\hat{a}\times\hat{b}) + x_2\hat{b}\cdot(\hat{a}\times\hat{b})$

$+x_3(\hat{a}\times\hat{b})\cdot(\hat{a}\times\hat{b}) = x_3$

$\Rightarrow \quad \vec{r} = (\vec{r}\cdot\hat{a})\hat{a} + (\vec{r}\cdot\hat{b})\hat{b} + (\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$

68. (1) $[\vec{a} + (\vec{a}\times\vec{b})\ \vec{b} + (\vec{a}\times\vec{b})\ \vec{a}\times\vec{b}]$

$= (\vec{a} + (\vec{a}\times\vec{b}))\cdot((\vec{b} + (\vec{a}\times\vec{b}))\times(\vec{a}\times\vec{b}))$

$= (\vec{a} + (\vec{a}\times\vec{b}))\cdot(\vec{b}\times(\vec{a}\times\vec{b}))$

$= (\vec{a} + (\vec{a}\times\vec{b}))\cdot(\vec{a} - (\vec{a}\cdot\vec{b})\vec{b})$

$= \vec{a}\cdot\vec{a} = 1 \qquad (\text{as } \vec{a}\cdot\vec{b} = 0,\ \vec{a}\cdot(\vec{a}\times\vec{b}) = 0)$

70. (4) $|\vec{a}| = 1, |\vec{b}| = 4, \vec{a}\cdot\vec{b} = 2$

$\vec{c} = (2\vec{a}\times\vec{b}) - 3\vec{b}$

or $\quad \vec{c} + 3\vec{b} = 2\vec{a}\times\vec{b}$

$\because \quad \vec{a}\cdot\vec{b} = 2$

$\Rightarrow \quad |\vec{a}|\cdot|\vec{b}|\cos\theta = 2$

or $\quad \cos\theta = \dfrac{2}{|\vec{a}|\cdot|\vec{b}|} = \dfrac{2}{4}$

$= \dfrac{1}{2}$

or $\quad \theta = \dfrac{\pi}{3}$

$\Rightarrow \quad |\vec{c} + 3\vec{b}|^2 = |2\vec{a}\times\vec{b}|^2$

or $\quad |\vec{c}|^2 + 9|\vec{b}|^2 + 2\vec{c}\cdot3\vec{b} = 4|\vec{a}|^2|\vec{b}|^2\sin^2\theta$

or $\quad |\vec{c}|^2 + 144 + 6\vec{b}\cdot\vec{c} = 48$

or $\quad |\vec{c}|^2 + 96 + 6(\vec{b}\cdot\vec{c}) = 0 \qquad$ (i)

or $\quad \vec{c} = 2\vec{a}\times\vec{b} - 3\vec{b}$

$\Rightarrow \quad \vec{b}\cdot\vec{c} = 0 - 3\times16$

$= -48$

Putting value of $b\,\vec{b}\cdot\vec{c}$ in Eq. (i), we have

$|\vec{c}|^2 + 96 - 6\times48$

$= 0 = 48\times4 = 192$

Again, putting the value of $|\vec{c}|$ in Eq. (i), we have

$192 + 96 + 6|\vec{b}|\cdot|\vec{c}|\cos\alpha = 0$

or $\quad 6\times4\times8\sqrt{3}\ \cos\alpha = -288$

or $\quad \cos\alpha = -\dfrac{288}{6\times4\times8\sqrt{3}} = -\dfrac{3}{2\sqrt{3}}$

$= -\dfrac{\sqrt{3}}{2}$

or $\quad \alpha = \dfrac{5\pi}{6}$

71. (4) $((\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c})$

$= (\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c}) + (\vec{a}\times\vec{c})\times(\vec{b}\times\vec{c})$

$= ((\vec{a}\times\vec{b})\cdot\vec{c})\vec{b} - ((\vec{a}\times\vec{b})\cdot\vec{b})\vec{c}$

$+ ((\vec{a}\times\vec{c})\cdot\vec{c})\vec{b} - ((\vec{a}\times\vec{c})\cdot\vec{b})\vec{c}$

$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{b} + \vec{c})$

$\Rightarrow \quad ((\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c}))\cdot(\vec{b} - \vec{c})$

$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{b} + \vec{c}).(\vec{b} - \vec{c})$

$= [\vec{a}\ \vec{b}\ \vec{c}](|\vec{b}|^2 - |c|^2) = 0$

72. (1) $\vec{a}\times\vec{b} = \vec{c}$

or $\quad \vec{a}\times(\vec{a}\times\vec{b}) = \vec{a}\times\vec{c}$

or $\quad (\vec{a}\cdot\vec{b})\vec{a} - |\vec{a}|^2\vec{b} = \vec{a}\times\vec{c}$

or $\quad \vec{b} = \dfrac{\beta\vec{a} - \vec{a}\times\vec{c}}{|\vec{a}|^2} \qquad (\because \vec{a}\cdot\vec{b} = \beta)$

73. (2) Taking dot product of $a(\vec{\alpha}\times\vec{\beta}) + b(\vec{\beta}\times\vec{\gamma}) + c(\vec{\gamma}\times\vec{\alpha}) = 0$ with $\vec{\gamma}$, $\vec{\alpha}$ and $\vec{\beta}$, respectively, we have

$a[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = 0$

$$b[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = 0$$

$$c[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = 0$$

Since at least one of a, b and $c \neq 0$, we have

$$[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = 0$$

Hence, $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ are coplanar.

74. (3) $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}) = \vec{b}$

or $[\vec{a}\,\vec{b}\,\vec{c}]\vec{b} = \vec{b}$

or $[\vec{a}\,\vec{b}\,\vec{c}] = 1$

Therefore, $\vec{a}$, $\vec{b}$ and $\vec{c}$ cannot be coplanar.

75. (3) Any vector $\vec{r}$ can be represented in terms of three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as

$$\vec{r} = x(\vec{a} \times \vec{b}) + y(\vec{b} \times \vec{c}) + z(\vec{c} \times \vec{a}) \text{ (i)}$$

Taking dot product with $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively, we have

$$x = \frac{\vec{r} \cdot \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ y = \frac{\vec{r} \cdot \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} \text{ and } z = \frac{\vec{r} \cdot \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

From (i), we have

$$[\vec{a}\,\vec{b}\,\vec{c}]\vec{r} = \frac{1}{2}(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$$

$\therefore$ Area of ΔABC

$$= \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

$$= |[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}|$$

76. (1) Differentiate the curve

$$6x + 8(xy_1 + y) + 4yy_1 = 0$$

m_T at $(1, 0)$ is $6 + 8(y_1(0)) = 0$

$$y_1(0) = -\frac{3}{4}$$

$$m_N = \frac{4}{3}$$

$$\text{Unit vector} = \pm\frac{(3\hat{i} + 4\hat{j})}{5}$$

Again normal vector of magnitude $10 = \pm(6\hat{i} + 8\hat{j})$

77. (1) $\{\vec{a} \times (\vec{b} + \vec{a} \times \vec{b})\} \cdot \vec{b}$

$$= \{\vec{a} \times \vec{b} + \vec{a} \times (\vec{a} \times \vec{b})\} \cdot \vec{b}$$

$$= [\vec{a}\,\vec{b}\,\vec{b}] + \{(\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b}\} \cdot \vec{b}$$

$$= 0 + (\vec{a} \cdot \vec{b})^2 - (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b})$$

$$= \cos^2\frac{\pi}{3} - 1 = -\frac{3}{4}$$

78. (1) $\vec{r} \times \vec{a} = \lambda\vec{a} + \mu\vec{b} + \gamma\vec{a} \times \vec{b}$

$\therefore$ $[\vec{r}\,\vec{a}\,\vec{a}] = \lambda\vec{a} \cdot \vec{a} + \mu\vec{b} \cdot \vec{a} + \gamma[\vec{a}\,\vec{b}\,\vec{a}]$

$$0 = \lambda|\vec{a}|^2 + 0 + 0$$

$$\lambda = 0$$

Also $[\vec{r}\,\vec{a}\,\vec{b}] = \lambda\vec{a} \cdot \vec{b} + \mu\vec{b} \cdot \vec{b} + \gamma[\vec{a}\,\vec{b}\,\vec{b}] = \mu$

Also $(\vec{r} \times \vec{a}) \times \vec{b} = \gamma(\vec{a} \times \vec{b}) \times \vec{b}$

or $(\vec{r} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{b})\vec{r} = \gamma\{(\vec{a} \cdot \vec{b})\vec{b} - (\vec{b} \cdot \vec{b})\vec{a}\}$

or $(\vec{r} \cdot \vec{b})\vec{a} = -\gamma\vec{a}$ or $\gamma = -(\vec{r} \cdot \vec{b})$

79. (3) The given equation reduces to

$$[\vec{a}\,\vec{b}\,\vec{c}]^2x^2 + 2[\vec{a}\,\vec{b}\,\vec{c}]x + 1 = 0 \Rightarrow D = 0$$

80. (2) $\vec{x} + \vec{c} \times \vec{y} = \vec{a}$ (i)

$$\vec{y} + \vec{c} \times \vec{x} = \vec{b} \quad \text{(ii)}$$

Taking cross with $\vec{c}$, we have

$$\vec{c} \times \vec{y} + \vec{c} \times (\vec{c} \times \vec{x}) = \vec{c} \times \vec{b}$$

$\Rightarrow$ $(\vec{a} - \vec{x}) + (\vec{c} \cdot \vec{x})\vec{c} - (\vec{c} \cdot \vec{c})\vec{x} = \vec{c} \times \vec{b}$

Also $\vec{x} + \vec{c} \times \vec{y} = \vec{a}$

$\Rightarrow$ $\vec{c} \cdot \vec{x} + \vec{c} \cdot (\vec{c} \times \vec{y}) = \vec{c} \cdot \vec{a}$

or $\vec{c} \cdot \vec{x} + 0 = \vec{c} \cdot \vec{a}$

or $\vec{c} \cdot \vec{x} = \vec{c} \cdot \vec{a}$

$\Rightarrow$ $\vec{a} - \vec{x} + (\vec{c} \cdot \vec{a})\vec{c} - (\vec{c} \cdot \vec{c})\vec{x} = \vec{c} \times \vec{b}$

or $\vec{x}(1 + (\vec{c} \cdot \vec{c})) = \vec{b} \times \vec{c} + \vec{a} + (\vec{c} \cdot \vec{a}) \cdot \vec{c}$

or $\vec{x} = \dfrac{\vec{b} \times \vec{c} + \vec{a} + (\vec{c} \cdot \vec{a})\vec{c}}{1 + \vec{c} \cdot \vec{c}}$

Similarly on taking cross product of Eq. (i), we find

$$\vec{y} = \frac{\vec{a} \times \vec{c} + \vec{b} + (\vec{c} \cdot \vec{b})\vec{c}}{1 + \vec{c} \cdot \vec{c}}$$

81. (3) $\vec{r} \times \vec{a} = \vec{b}$

or $\vec{d} \times (\vec{r} \times \vec{a}) = \vec{d} \times \vec{b}$

or $(\vec{a}\cdot\vec{d})\vec{r}-(\vec{d}\cdot\vec{r})\vec{a}=\vec{d}\times\vec{b}$ (i)

$\vec{r}\times\vec{c}=\vec{d}$

or $\vec{b}\times(\vec{r}\times\vec{c})=\vec{b}\times\vec{d}$

or $(\vec{b}\cdot\vec{c})\vec{r}-(\vec{b}\cdot\vec{r})\vec{c}=\vec{b}\times\vec{d}$ (ii)

Adding (i) and (ii), we get

$$(\vec{a}\cdot\vec{d}+\vec{b}\cdot\vec{c})\vec{r}-(\vec{d}\cdot\vec{r})\vec{a}-(\vec{b}\cdot\vec{r})\vec{c}=\vec{0}$$

Now $\vec{r}\cdot\vec{d}=0$ and $\vec{b}\cdot\vec{r}=0$ as $\vec{d}$ and $\vec{r}$ as well as $\vec{b}$ and $\vec{r}$ are mutually perpendicular.

Hence, $(\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{d})\vec{r}=\vec{0}$

82. (2) Let $\vec{a}\times\vec{b}=x\hat{i}+y\hat{j}+z\hat{k}$. Therefore,

$$[\vec{a}\,\vec{b}\,\hat{i}]=(\vec{a}\times\vec{b})\cdot\hat{i}=x$$
$$[\vec{a}\,\vec{b}\,\hat{j}]=(\vec{a}\times\vec{b})\cdot\hat{j}=y$$
$$[\vec{a}\,\vec{b}\,\hat{k}]=(\vec{a}\times\vec{b})\cdot\hat{k}=z$$

Hence, $[\vec{a}\,\vec{b}\,\hat{i}]\hat{i}+[\vec{a}\,\vec{b}\,\hat{j}]\hat{j}$

$$=[\vec{a}\,\vec{b}\,\hat{k}]\hat{k}=x\hat{i}+y\hat{j}+z\hat{k}=\vec{a}\times\vec{b}$$

83. (1) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$$=5(\hat{i}+2\hat{j}+2\hat{k})-6(\hat{i}+\hat{j}+2\hat{k})$$

$\Rightarrow$ $(1+\alpha)\hat{i}+\beta(1+\alpha)\hat{j}+\gamma(1+\alpha)(1+\beta)\hat{k}=-\hat{i}+4\hat{j}-2\hat{k}$

$\Rightarrow$ $1+\alpha=-1$, $\beta=-4$ and $\gamma(-1)(-3)=-2$

$\Rightarrow$ $\gamma=-\dfrac{2}{3}$

84. (2) If $\vec{a}(x)$ and $\vec{b}(x)$ are $\perp$, then $\vec{a}\cdot\vec{b}=0$

$\Rightarrow$ $\sin x\cos 2x+\cos x\sin 2x=0$

$\sin(3x)=0=\sin 0$

$3x=n\pi$ or $x=\dfrac{n\pi}{3}$

Therefore, the two vectors are $\perp$ for infinite values of x.

85. (2) $(\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i})=\begin{vmatrix}\vec{a}\cdot\vec{b} & \vec{a}\cdot\hat{i}\\ \vec{b}\cdot\hat{i} & \hat{i}\cdot\hat{i}\end{vmatrix}$

$$=(\vec{a}\cdot\vec{b})-(\vec{a}\cdot\hat{i})(\vec{b}\cdot\hat{i})$$

Similarly, $(\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j})=(\vec{a}\cdot\vec{b})-(\vec{a}\cdot\hat{j})(\vec{b}\cdot\hat{j})$

and $(\vec{a}\times\hat{k})\cdot(\vec{b}\times\hat{k})=\vec{a}\cdot\vec{b}-(\vec{a}\cdot\hat{k})(\vec{b}\cdot\hat{k})$

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$.

$\therefore$ $(\vec{a}\cdot\hat{i})=a_1$, $\vec{a}\cdot\hat{j}=a_2$,

$\vec{a}\cdot\hat{k}=a_3$, $\vec{b}\cdot\hat{i}=b_1$, $\vec{b}\cdot\hat{j}=b_2$, $(\vec{b}\cdot\hat{k})=b_3$

$\Rightarrow$ $(\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i})+(\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j})+(\vec{a}\times\hat{k})\cdot(\vec{b}\times\hat{k})$

$$=3\vec{a}\cdot\vec{b}-(a_1b_1+a_2b_2+a_3b_3)$$
$$=3\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{b}=2\vec{a}\cdot\vec{b}$$

86. (2) $(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})$

$$=((\vec{a}\times\vec{b})\cdot\vec{c})\vec{r}-((\vec{a}\times\vec{b})\cdot\vec{r})\cdot\vec{c}$$
$$=[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}$$

Similarly, $(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})$

$$=[\vec{b}\,\vec{c}\,\vec{a}]\vec{r}-[\vec{b}\,\vec{c}\,\vec{r}]\vec{a}$$

and $(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})=[\vec{c}\,\vec{a}\,\vec{b}]\vec{r}-[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}$

$\Rightarrow$ $(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})+(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})+(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})$

$$=3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-([\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c})$$
$$=3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}=2[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$$

87. (1) We have,

$$\vec{a}\cdot\vec{p}=\vec{a}\cdot\frac{(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}=1$$

$$\vec{a}\cdot\vec{q}=\vec{a}\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{[\vec{a}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}=0$$

Similarly, $\vec{a}\cdot\vec{r}=0$, $\vec{b}\cdot\vec{p}=0$, $\vec{b}\cdot\vec{q}=1$, $\vec{b}\cdot\vec{r}=0$,

$\vec{c}\cdot\vec{p}=0$, $\vec{c}\cdot\vec{q}=0$ and $\vec{c}\cdot\vec{r}=1$

$\therefore$ $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{p}+\vec{q}+\vec{r})$

$$= \vec{a}\cdot\vec{p} + \vec{a}\cdot\vec{q} + \vec{a}\cdot\vec{r}$$
$$+ \vec{b}\cdot\vec{p} + \vec{b}\cdot\vec{q} + \vec{b}\cdot\vec{r}$$
$$+ \vec{c}\cdot\vec{p} + \vec{c}\cdot\vec{q} + \vec{c}\cdot\vec{r}$$
$$= 1 + 1 + 1 = 3$$

88. (2) A vector perpendicular to the plane of $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ is

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}.$$

Now for any point $R(\vec{r})$ in the plane of A, B and C is

$$(\vec{r} - \vec{a}) \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) = 0$$
$$\vec{r} \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$$
$$- \vec{a} \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) = 0$$
$$\vec{r} \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$$
$$= \vec{0} + \vec{a} \cdot \vec{b} \times \vec{c} + \vec{0}$$
$$\vec{r} \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]$$

89. (3) Given that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar. Thus,

$$[\vec{a}\,\vec{b}\,\vec{c}] \neq 0 \qquad \text{(i)}$$

Again $\vec{a} \times (\vec{b} \times \vec{c}) \cdot (\vec{a} \times \vec{c}) = 0$

or $[(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}] \cdot (\vec{a} \times \vec{c}) = 0$

or $(\vec{a} \cdot \vec{c})[\vec{b}\,\vec{a}\,\vec{c}] = 0$

or $(\vec{a} \cdot \vec{c}) = 0$

Hence, $\vec{a}$ and $\vec{c}$ are perpendicular. (ii)

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

or $[\vec{a} \times (\vec{b} \times \vec{c})] \times \vec{c} = \vec{0}$

90. (3) Consider a tetrahedron with vertices $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

$$\text{Volume } V = \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$$

Now centroids of the faces OAB, OAC, OBC and ABC are $G_1(a/3, b/3, 0)$, $G_2(a/3, 0, c/3)$, $G_3(0, b/3, c/3)$ and $G_4(a/3, b/3, c/3)$, respectively.

$$\overrightarrow{G_4G_1} = \vec{c}/3,\ \overrightarrow{G_4G_2} = \vec{b}/3,\ \overrightarrow{G_4G_3} = \vec{a}/3.$$

Volume of tetrahedron by centroids

$$V' = \frac{1}{6}\begin{bmatrix} \frac{\vec{a}}{3} & \frac{\vec{b}}{3} & \frac{\vec{c}}{3} \end{bmatrix} = \frac{1}{27}V$$

$$\Rightarrow \quad K = 27$$

91. (3) $[(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})\ \ (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})\ \ (\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})]$

$$= [[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{b}\ \ [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}\ \ [\vec{a}\,\vec{b}\,\vec{c}]\cdot\vec{a}]$$
$$= [\vec{a}\,\vec{b}\,\vec{c}]^3\ [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^4$$

92. (4) $\vec{r} = x_1(\vec{a} \times \vec{b}) + x_2(\vec{b} \times \vec{c}) + x_3(\vec{c} \times \vec{a})$

$$\Rightarrow \quad \vec{r} \cdot \vec{a} = x_2[\vec{a}\,\vec{b}\,\vec{c}],\ \vec{r} \cdot \vec{b} = x_3[\vec{b}\,\vec{c}\,\vec{a}]$$

and $\vec{r} \cdot \vec{c} = x_1[\vec{c}\,\vec{a}\,\vec{b}] = x_1[\vec{a}\,\vec{b}\,\vec{c}]$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 4\vec{r} \cdot (\vec{a} + \vec{b} + \vec{c})$$

93. (1) Let $\vec{v} = x\vec{a} + y\vec{b} + z\vec{a} \times \vec{b}$

Given: $\vec{a} \cdot \vec{b} = 0,\ \vec{v} \cdot \vec{a} = 0,\ \vec{v} \cdot \vec{b} = 1,\ [\vec{v}\,\vec{a}\,\vec{b}] = 1$

$$\Rightarrow \vec{v} \cdot \vec{a} = x\,\vec{a} \cdot \vec{a} = x|\vec{a}|$$
$$(\because \vec{a} \cdot \vec{b} = 0,\ \vec{a} \cdot \vec{a} \times \vec{b} = 0)$$

$$\Rightarrow \quad x = 0$$

Again, $\vec{v} \cdot \vec{b} = y|\vec{b}|^2$ or $1 = yb^2$

$$\therefore \quad y = \frac{1}{b^2} \qquad \text{(ii)}$$

Again, $\vec{v} \cdot (\vec{a} \times \vec{b}) = z(\vec{a} \times \vec{b})^2$

$$\Rightarrow \quad 1 = z(\vec{a} \times \vec{b})^2 \quad \text{or} \quad z = \frac{1}{|\vec{a} \times \vec{b}|^2}$$

Hence, $\vec{v} = \frac{1}{|\vec{b}|^2}\vec{b} + \frac{1}{|\vec{a} \times \vec{b}|^2}\vec{a} \times \vec{b}$

94. (4) Volume of the parallelepiped formed by $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ is 4.

Therefore, the volume of the parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $\frac{1}{4}$.

$$\vec{b} \times \vec{c} = [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{a}' = \frac{1}{4}\vec{a}'$$

$$|\vec{b}\times\vec{c}| = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$$

Length of altitude $= \frac{1}{4}\times 2\sqrt{2} = \frac{1}{\sqrt{2}}$

95. (4) $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{\hat{i}+\hat{j}-\hat{k}}{2}$

Multiple Correct Answers Type

1. (1), (2)

We have $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}\cdot\vec{b})$

or $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos 2\theta$

or $|\vec{a}-\vec{b}|^2 = 2-2\cos 2\theta \quad (\because |\vec{a}| = |\vec{b}| = 1)$

$= 4\sin^2\theta$

or $|\vec{a}-\vec{b}| = 2|\sin\theta|$

Now, $|\vec{a}-\vec{b}| < 1$

$\Rightarrow \quad 2|\sin\theta| < 1$

or $|\sin\theta| < \frac{1}{2}$

$\Rightarrow \quad \theta \in [0, \pi/6)$ or $\theta \in (5\pi/6, \pi]$

2. (1), (3)

$\vec{a}\times(\vec{b}\times\vec{c}) + (\vec{a}\cdot\vec{b})\vec{b}$

$= (4-2x-\sin y)\vec{b} + (x^2-1)\vec{c}$

or $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{b}$

$= (4-2x-\sin y)\vec{b} + (x^2-1)\vec{c}$

Now, $(\vec{c}\cdot\vec{c})\vec{a} = \vec{c}$. Therefore,

$(\vec{c}\cdot\vec{c})(\vec{a}\cdot\vec{c}) = (\vec{c}\cdot\vec{c})$ or $\vec{a}\cdot\vec{c} = 1$

$\Rightarrow \quad 1 + \vec{a}\cdot\vec{b} = 4-2x-\sin y, \quad x^2-1 = -(\vec{a}\cdot\vec{b})$

or $1 = 4 - 2x - \sin y + x^2 - 1$

or $\sin y = x^2 - 2x + 2 = (x-1)^2 + 1$

But $\sin y \le 1 \Rightarrow x = 1, \sin y = 1$

$\Rightarrow \quad y = (4n+1)\dfrac{\pi}{2}, \quad n \in I$

3. (1), (2), (3), (4)

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors inclined at an angle θ, we have

$|\vec{a}| = |\vec{b}| = 1$ and $\cos\theta = \vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$

Now, $\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})$ (i)

$\Rightarrow \quad \vec{a}\cdot\vec{c} = \alpha(\vec{a}\cdot\vec{a}) + \beta(\vec{a}\cdot\vec{b}) + \gamma\{\vec{a}\cdot(\vec{a}\times\vec{b})\}$

$\Rightarrow \quad \cos\theta = \alpha|\vec{a}|^2 \quad (\because \vec{a}\cdot\vec{b} = 0,\ \vec{a}\cdot(\vec{a}\times\vec{b}) = 0)$

$= \alpha$

Similarly, by taking dot product on both sides of (i) by $\vec{b}$, we get $\beta = \cos\theta$

$\therefore \quad \alpha = \beta$

Again, $\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})$

$\Rightarrow \quad |\vec{c}|^2 = |\alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})|^2$

$= \alpha^2|\vec{a}|^2 + \beta^2|\vec{b}|^2 + \gamma^2|\vec{a}\times\vec{b}|^2$

$+ 2\alpha\beta(\vec{a}\cdot\vec{b}) + 2\alpha\gamma\{\vec{a}\cdot(\vec{a}\times\vec{b})\}$

$+ 2\beta\gamma(\vec{b}\cdot\{\vec{a}\times\vec{b}\})$

$\Rightarrow \quad 1 = \alpha^2 + \beta^2 + \gamma^2|\vec{a}\times\vec{b}|^2$

$= 2\alpha^2 + \gamma^2\{|\vec{a}|^2|\vec{b}|^2\sin^2\pi/2\}$

$= 2\alpha^2 + \gamma^2$ or $\alpha^2 = \dfrac{1-\gamma^2}{2}$

But $\alpha = \beta = \cos\theta$.

$1 = 2\alpha^2 + \gamma^2 \Rightarrow \gamma^2 = 1 - 2\cos^2\theta = -\cos 2\theta$

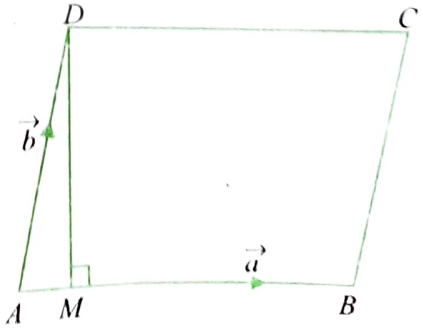
$\therefore \quad \beta^2 = \dfrac{1-\gamma^2}{2} = \dfrac{1+\cos 2\theta}{2}$

4. (1), (2), (3)

We have

AM = projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$\therefore \quad \overrightarrow{AM} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$



Now, in ΔADM

$$\overrightarrow{AD} = \overrightarrow{AM} + \overrightarrow{MD} \text{ or } \overrightarrow{DM} = \overrightarrow{AM} - \overrightarrow{AD}$$

$$\Rightarrow \quad \overrightarrow{DM} = \frac{(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2} - \vec{b}$$

$$= \frac{1}{|\vec{a}|^2}[(\vec{a}\cdot\vec{b})\vec{a} - |\vec{a}|^2\vec{b}]$$

$$\Rightarrow \quad \overrightarrow{MD} = \frac{1}{|\vec{a}|^2}[|\vec{a}|^2\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}]$$

Now, $\dfrac{\vec{a}\times(\vec{a}\times\vec{b})}{|\vec{a}|^2} = \dfrac{1}{|\vec{a}|^2}[(\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{a})\vec{b}]$

$$= \overrightarrow{DM}$$

5. (1), (3)

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

and $(\vec{a}\times\vec{b})\times\vec{c} = -(\vec{c}\cdot\vec{b})\vec{a} + (\vec{a}\cdot\vec{c})\vec{b}$

We have been given

$$(\vec{a}\times(\vec{b}\times\vec{c}))\cdot((\vec{a}\times\vec{b})\times\vec{c}) = 0$$

$$\therefore \quad ((\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c})\cdot((\vec{a}\cdot\vec{c})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}) = 0$$

or $(\vec{a}\cdot\vec{c})^2|\vec{b}|^2 - (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{b})$

$$-(\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{c}) + (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a}) = 0$$

or $(\vec{a}\cdot\vec{c})^2|\vec{b}|^2 = (\vec{a}\cdot\vec{c})(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$

or $(\vec{a}\cdot\vec{c})((\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})) = 0$

$\vec{a}\cdot\vec{c} = 0$ or $(\vec{a}\cdot\vec{c})|\vec{b}|^2 = (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$

6. (1), (3)

We have $[\vec{p}\ \vec{q}\ \vec{r}] = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$. Therefore,

$$[\vec{p}\ \vec{q}\ \vec{r}] > 0$$

(1) $x > 0, x[\vec{a}\ \vec{b}\ \vec{c}] + \dfrac{[\vec{p}\ \vec{q}\ \vec{r}]}{x} \geq 2$

(using A.M. $\geq$ G.M.)

(2) Similarly, use A.M. $\geq$ G.M.

7. (1), (2), (3), (4)

$$a_1 + a_2\cos 2x + a_3\sin^2 x = 0 \ \forall\ x \in R$$

or $(a_1 + a_2) + \sin^2 x\,(a_3 - 2a_2) = 0$

$$\Rightarrow \quad a_1 + a_2 = 0 \text{ and } a_3 - 2a_2 = 0$$

$$\frac{a_1}{-1} = \frac{a_2}{1} = \frac{a_3}{2} = \lambda\ (\neq 0)$$

$$\Rightarrow \quad a_1 = -\lambda,\ a_2 = \lambda,\ a_3 = 2\lambda$$

8. (1), (2), (3), (4)

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

or $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

or $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$ (i)

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|} \quad \text{(ii)}$$

From (i) and (ii),

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$$

If $\theta = \pi/4$, then $\sin\theta = \cos\theta = 1/\sqrt{2}$. Therefore,

$$|\vec{a}\times\vec{b}| = \frac{|\vec{a}||\vec{b}|}{\sqrt{2}} \text{ and } \vec{a}\cdot\vec{b} = \frac{|\vec{a}||\vec{b}|}{\sqrt{2}}$$

$$|\vec{a}\times\vec{b}| = \vec{a}\cdot\vec{b}$$

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n} = \frac{|\vec{a}||\vec{b}|}{\sqrt{2}}\hat{n}$$

$$= (\vec{a}\cdot\vec{b})\hat{n}$$

9. (1), (2), (3), (4)

Since $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar,

$$\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})$$

$$\therefore \quad \vec{r}\times\vec{b} = \vec{a}$$

$$\Rightarrow \quad x\vec{a}\times\vec{b} + z\{(\vec{a}\cdot\vec{b})\vec{b} - (\vec{b}\cdot\vec{b})\vec{a}\} = \vec{a}$$

or $-(1 + z|\vec{b}|^2)\vec{a} + x\vec{a}\times\vec{b} = 0$

(since $\vec{a}\cdot\vec{b} = 0$)

$$\therefore \quad x = 0 \text{ and } z = -\frac{1}{|\vec{b}|^2}$$

Thus, $\vec{r} = y\vec{b} - \dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$, where y is the parameter.

10. **(2), (4)**

Since $\vec{a} = (1, 3, \sin 2\alpha)$ makes an obtuse angle with the z-axis, its z-component is negative. Thus,

$$-1 \le \sin 2\alpha < 0 \qquad \text{(i)}$$

But $\vec{b} \cdot \vec{c} = 0$ ($\because$ orthogonal)

$$\tan^2 \alpha - \tan \alpha - 6 = 0$$

$$\therefore \quad (\tan \alpha - 3)(\tan \alpha + 2) = 0$$

$$\Rightarrow \quad \tan \alpha = 3, -2$$

Now, $\tan \alpha = 3$. Therefore,

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{6}{1+9} = \frac{3}{5}$$

(not possible as $\sin 2\alpha < 0$)

Now, if $\tan \alpha = -2$,

$$\Rightarrow \quad \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{-4}{1+4} = \frac{-4}{5}$$

$$\Rightarrow \quad \tan 2\alpha > 0$$

Hence, 2α is the third quadrant. Also, $\sqrt{\sin \alpha/2}$ is meaningful. If $0 < \sin \alpha/2 < 1$, then

$$\alpha = (4n+1)\pi - \tan^{-1} 2$$

and $\alpha = (4n+2)\pi - \tan^{-1} 2$.

11. **(2), (4)**

$$\vec{a} \times (\vec{r} \times \vec{a}) = \vec{a} \times \vec{b}$$

$$3\vec{r} - (\vec{a} \cdot \vec{r})\vec{a} = \vec{a} \times \vec{b}$$

Also $|\vec{r} \times \vec{a}| = |\vec{b}|$

$$\Rightarrow \quad \sin^2 \theta = \frac{2}{3}$$

or $\quad (1 - \cos^2 \theta) = \frac{2}{3}$

or $\quad \frac{1}{3} = \cos^2 \theta$

$$\Rightarrow \quad \vec{a} \cdot \vec{r} = \pm 1$$

$$\Rightarrow \quad 3\vec{r} \pm \vec{a} = \vec{a} \times \vec{b}$$

or $\quad \vec{r} = \frac{1}{3}(\vec{a} \times \vec{b} \pm \vec{a})$

12. **(2), (4)**

$$(\vec{a} - \vec{b}) \times [(\vec{b} + \vec{a}) \times (2\vec{a} + \vec{b})] = \vec{b} + \vec{a}$$

or $\quad \{(\vec{a} - \vec{b}) \cdot (2\vec{a} + \vec{b})\}(\vec{b} + \vec{a})$

$$-\{(\vec{a} - \vec{b}) \cdot (\vec{b} + \vec{a})\}(2\vec{a} + \vec{b}) = \vec{b} + \vec{a}$$

or $\quad (2 - \vec{a} \cdot \vec{b} - 1)(\vec{b} + \vec{a}) = \vec{b} + \vec{a}$

$$\Rightarrow \quad \text{either } \vec{b} + \vec{a} = \vec{0} \text{ or } 1 - \vec{a} \cdot \vec{b} = 1$$

$$\Rightarrow \quad \text{either } \vec{b} = -\vec{a} \text{ or } \vec{a} \cdot \vec{b} = 0$$

$$\Rightarrow \quad \text{either } \theta = \pi \text{ or } \theta = \pi/2$$

13. **(1), (4)**

Given $\vec{c} = \lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3(\vec{a} \times \vec{b})$ (i)

and $\vec{a} \cdot \vec{b} = 0, |\vec{a}| = 1, |\vec{b}| = 1$

From (i), $\vec{a} \cdot \vec{c} = \lambda_1, \vec{c} \cdot \vec{b} = \lambda_2$

and $\vec{c} \cdot (\vec{a} \times \vec{b}) = |\vec{a} \times \vec{b}|^2 \lambda_3$

$$= (1 \cdot 1 \sin 90°)^2 \lambda_3 = \lambda_3$$

Hence, $\lambda_1 + \lambda_2 + \lambda_3 = (\vec{a} + \vec{b} + \vec{a} \times \vec{b}) \cdot \vec{c}$

14. **(2), (3), (4)**

Obviously, $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$ is a vector in the plane of $\vec{a}$ and $\vec{b}$ and hence perpendicular to $\vec{a} \times \vec{b}$. It is also equally inclined to $\vec{a}$ and $\vec{b}$ as it is along the angle bisector.

15. **(1), (4)**

$$|\vec{a} + \vec{b}| = |\vec{a} - 2\vec{b}|$$

$$\Rightarrow \quad \vec{a} \cdot \vec{b} = \frac{|\vec{b}|^2}{2}$$

Also $\quad \vec{a} \cdot \vec{b} + \frac{1}{|\vec{b}|^2 + 2}$

$$= \frac{|\vec{b}|^2 + 2}{2} + \frac{1}{|\vec{b}|^2 + 2} - 1$$

$$\ge \sqrt{2} - 1 \qquad \text{(using A.M. } \ge \text{ G.M.)}$$

16. **(2), (4)**

$$\vec{V}_1 = \vec{V}_2$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$$

or $\quad (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

or $\quad (\vec{a} \cdot \vec{b})\vec{c} = (\vec{b} \cdot \vec{c})\vec{a}$

Thus, either $\vec{c}$ and $\vec{a}$ are collinear or $\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{c} \Rightarrow \vec{b} = \lambda(\vec{a} \times \vec{c})$

17. (2), (3)

We have $\vec{A} + \vec{B} = \vec{a}$

or $\vec{A}\cdot\vec{a} + \vec{B}\cdot\vec{a} = \vec{a}\cdot\vec{a}$

or $1 + \vec{B}\cdot\vec{a} = a^2$ (given $\vec{A}\cdot\vec{a} = 1$)

or $\vec{B}\cdot\vec{a} = a^2 - 1$ (i)

Also $\vec{A} \times \vec{B} = \vec{b}$

or $\vec{a} \times (\vec{A} \times \vec{B}) = \vec{a} \times \vec{b}$

or $(\vec{a}\cdot\vec{B})\vec{A} - (\vec{a}\cdot\vec{A})\vec{B} = \vec{a} \times \vec{b}$

or $(a^2 - 1)\vec{A} - \vec{B} = \vec{a} \times \vec{b}$

(using (i) and $\vec{a}\cdot\vec{A} = 1$) (ii)

and $\vec{A} + \vec{B} = a$ (iii)

From (ii) and (iii), we have

$$\vec{A} = \frac{(\vec{a} \times \vec{b}) + \vec{a}}{a^2}$$

$$\vec{B} = \vec{a} - \left\{\frac{(\vec{a} \times \vec{b}) + \vec{a}}{a^2}\right\}$$

$$= \frac{(\vec{b} \times \vec{a}) + \vec{a}(a^2 - 1)}{a^2}$$

Thus, $\vec{A} = \dfrac{(\vec{a} \times \vec{b}) + \vec{a}}{a^2}$

and $\vec{B} = \dfrac{(\vec{b} \times \vec{a}) + \vec{a}(a^2 - 1)}{a^2}$

18. (3), (4)

Since $[\vec{a}\,\vec{b}\,\vec{c}\,] = 0$, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors.

Further, since $\vec{d}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$,

$$\vec{d}\cdot\vec{a} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = 0$$

$$\vec{d}\cdot\vec{x} = \vec{d}\cdot\vec{y} = \vec{d}\cdot\vec{z} = 0$$

$$\vec{d}\cdot\vec{r} = 0$$

19. (2), (4)

Let $\vec{\alpha} = \hat{i} - \hat{j} - \hat{k}$, $\vec{\beta} = \hat{i} + \hat{j} + \hat{k}$

and $\vec{\gamma} = -\hat{i} + \hat{j} + \hat{k}$.

Let required vector $\vec{a} = x\hat{i} + y\hat{j} + z\hat{j}$.

Then $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are coplanar, i.e.,

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 0 \text{ or } y = z$$

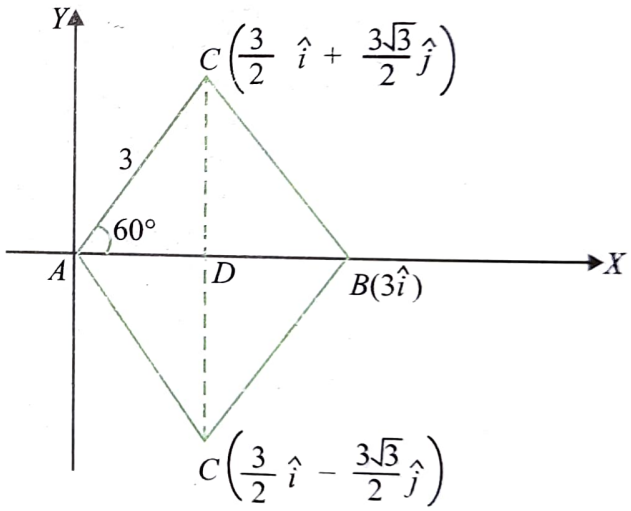
Also, $\vec{a}$ and $\vec{\alpha}$ are perpendicular

$\Rightarrow \quad x - y - z = 0$

$\Rightarrow \quad x = zy$

Hence, options (2) and (4) are correct.

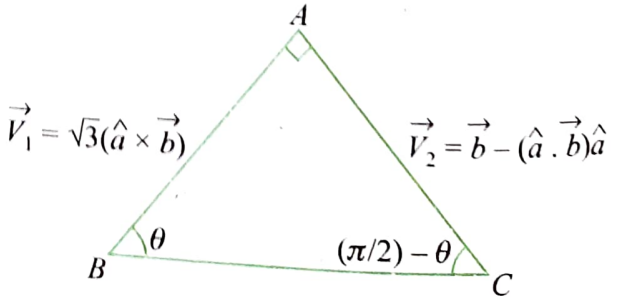
20. (2), (4)



21. (1), (2), (3)

Consider $\vec{V}_1\cdot\vec{V}_2 = 0 \Rightarrow A = 90°$

Using the sine law, $\left|\dfrac{\vec{b} - (\hat{a}\cdot\vec{b})\hat{a}}{\sin\theta}\right| = \dfrac{\sqrt{3}\,|\hat{a} \times \vec{b}|}{\cos\theta}$



or $\tan\theta = \dfrac{1}{\sqrt{3}} \dfrac{|\vec{b} - (\hat{a}\cdot\vec{b})\hat{a}|}{|\hat{a} \times \vec{b}|}$

$$= \frac{1}{\sqrt{3}} \frac{|(\hat{a} \times \vec{b}) \times \hat{a}|}{|\hat{a} \times \vec{b}|}$$

$$= \frac{1}{\sqrt{3}} \frac{|\hat{a} \times \vec{b}||\hat{a}| \sin 90°}{|\hat{a} \times \vec{b}|} = \frac{1}{\sqrt{3}}$$

or $\theta = \frac{\pi}{6}$

22. (1), (2)

Given, $\frac{1}{6}\hat{i} - \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k} = (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$

$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} \qquad \text{(i)}$$

[$\because$ $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar]

$$[\vec{a}\,\vec{b}\,\vec{d}] = (\vec{a} \times \vec{b}) \cdot \vec{d}$$

$$= |\vec{a} \times \vec{b}||\vec{d}| \cos\theta$$

($\because \vec{d} \perp \vec{a}, \vec{d} \perp \vec{b}, \ \therefore \vec{d} \parallel \vec{a} \times \vec{b}$)

$$= ab \sin 30° \cdot 1 \cdot (\pm 1) \quad (\because \theta = 0 \text{ or } \pi)$$

$$= 1 \cdot 1 \cdot \frac{1}{2} \cdot 1(\pm 1) = \pm\frac{1}{2}$$

From (i), we have

$$\vec{c} = \pm\left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \pm\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

23. (1), (2), (3)

We know that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Given $\vec{a} + 2\vec{b} + 3\vec{c} = \vec{0}$

or $2\vec{a} \times \vec{b} = 6\vec{b} \times \vec{c} = 3\vec{c} \times \vec{a}$

Hence $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$

$= 2(\vec{a} \times \vec{b})$ or $6(\vec{b} \times \vec{c})$ or $3(\vec{c} \times \vec{a})$

24. (1), (2)

$$\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\vec{b}$$

$$= \vec{a}(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})\vec{b}$$

$$= \vec{b} \times (\vec{a} \times \vec{b})$$

$$\Rightarrow \quad |\vec{u}| = |\vec{b} \times (\vec{a} \times \vec{b})|$$

$$= |\vec{b}||\vec{a} \times \vec{b}| \sin 90°$$

$$= |\vec{b}||\vec{a} \times \vec{b}|$$

$$= |\vec{v}|$$

Also $\vec{u} \cdot \vec{b} = \vec{b} \cdot \vec{b} \times (\vec{a} \times \vec{b})$

$$= [\vec{b}\,\vec{b}\,\vec{a} \times \vec{b}]$$

$$= 0$$

$$\Rightarrow \quad |\vec{v}| = |\vec{u}| + |\vec{u} \cdot \vec{b}|$$

25. (1), (3)

$\vec{a} \times \vec{b} = \vec{c}, \ \vec{b} \times \vec{c} = \vec{a}$

Taking cross with $\vec{b}$ in the first equation, we get

$$\vec{b} \times (\vec{a} \times \vec{b}) = \vec{b} \times \vec{c} = \vec{a}$$

or $|\vec{b}|^2 \vec{a} - (\vec{a} \cdot \vec{b})\vec{b} = \vec{a}$

$\Rightarrow \quad |\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 0$

Also $|\vec{a} \times \vec{b}| = |\vec{c}|$ or $|\vec{a} \parallel \vec{b}| \sin\frac{\pi}{2} = |\vec{c}|$

or $|\vec{a}| = |\vec{c}|$

26. (2), (4)

$$\vec{d} \cdot \vec{a} = [\vec{a}\,\vec{b}\,\vec{c}] \cos y = -\vec{d} \cdot (\vec{b} + \vec{c})$$

or $\cos y = -\frac{\vec{d} \cdot (\vec{b} + \vec{c})}{[\vec{a} \cdot \vec{b} \cdot \vec{c}]}$

Similarly, $\sin x = \frac{\vec{d} \cdot (\vec{a} + \vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\frac{\vec{d} \cdot (\vec{a} + \vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]} = -2$

$\therefore \quad \sin x + \cos y + 2 = 0$

or $\sin x + \cos y = -2$

or $\sin x = -1, \cos y = -1$

Since we want the minimum value of $x^2 + y^2$,

$x = -\pi/2, y = \pi$

Therefore, the minimum value of $x^2 + y^2$ is $5\pi^2/4$.

27. (2), (3)

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$$

or $(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = \frac{1}{2}\vec{b}$

$\Rightarrow \quad \vec{a}\cdot\vec{c} = \frac{1}{2}$ and $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow \quad 1\cdot 1\cos\alpha = \frac{1}{2}$ and $\vec{a} \perp \vec{b}$

$\Rightarrow \quad \alpha = \frac{\pi}{3}$ and $\vec{a} \perp \vec{b}$

28. (1), (2), (3)

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\overrightarrow{BC} = \frac{2\vec{u}}{|\vec{u}|} - \frac{\vec{u}}{|\vec{u}|} + \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u}}{|\vec{u}|} + \frac{\vec{v}}{|\vec{v}|}$$

$$\overrightarrow{AB}\cdot\overrightarrow{BC} = \left(\frac{\vec{u}}{|\vec{u}|} - \frac{\vec{v}}{|\vec{v}|}\right)\left(\frac{\vec{u}}{|\vec{u}|} + \frac{\vec{v}}{|\vec{v}|}\right)$$

$$= (\hat{u} - \hat{v})\cdot(\hat{u} + \hat{v}) = 1 - 1 = 0$$

$\Rightarrow \quad \angle B = 90°$

$\Rightarrow \quad 1 + \cos 2A + \cos 2B + \cos 2C = 0$

29. (1), (2), (3)

Let $\vec{A} = \vec{a}\times\vec{b}$, $\vec{B} = \vec{c}\times\vec{d}$ and $\vec{C} = \vec{e}\times\vec{f}$

We know that $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$= (\vec{a}\times\vec{b})\cdot[(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f})]$

$= (\vec{a}\times\vec{b})\cdot[\{(\vec{c}\times\vec{d})\cdot\vec{f}\}\vec{e} - \{(\vec{c}\times\vec{d})\cdot\vec{e}\}\vec{f}]$

$= [\vec{c}\,\vec{d}\,\vec{f}][\vec{a}\,\vec{b}\,\vec{e}] - [\vec{c}\,\vec{d}\,\vec{e}][\vec{a}\,\vec{b}\,\vec{f}]$

Similarly, other parts can be obtained.

30. (1), (3)

Here $(l\vec{a} + m\vec{b})\times\vec{b} = \vec{c}\times\vec{b}$

or $\quad l\vec{a}\times\vec{b} = \vec{c}\times\vec{b}$

or $\quad l(\vec{a}\times\vec{b})^2 = (\vec{c}\times\vec{b})\cdot(\vec{a}\times\vec{b})$

or $\quad l = \dfrac{(\vec{c}\times\vec{b})\cdot(\vec{a}\times\vec{b})}{(\vec{a}\times\vec{b})^2}$

Similarly, $m = \dfrac{(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{a})}{(\vec{b}\times\vec{a})^2}$

31. (2), (3), (4)

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\cdot(\vec{a}\times\vec{d}) = 0$

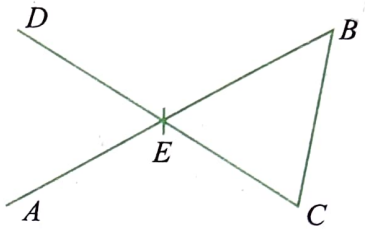
or $\quad ([\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a})\cdot(\vec{a}\times\vec{d}) = 0$

or $\quad [\vec{a}\,\vec{c}\,\vec{d}][\vec{b}\,\vec{a}\,\vec{d}] = 0$

Hence, either $\vec{c}$ or $\vec{b}$ must lie in the plane of $\vec{a}$ and $\vec{d}$.

32. (1), (2)

Let $\overrightarrow{EB} = p$, $\overrightarrow{AB}$ and $\overrightarrow{CE} = q\,\overrightarrow{CD}$.



Then $0 < p$ and $q \le 1$

Since $\overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CE} = \vec{0}$

$$pm(2\hat{i} - 6\hat{j} + 2\hat{k}) + (\hat{i} - 2\hat{j}) + qn(-6\hat{i} + 15\hat{j} - 3\hat{k}) = \vec{0}$$

or $\quad (2pm + 1 - 6qn)\hat{i} + (-6pm - 2 + 15qn)\hat{j} + (2pm - 6qn)\hat{k} = \vec{0}$

$\Rightarrow \quad 2pm - 6qn + 1 = \vec{0}$, $-6pm - 2 + 15qn = \vec{0}$, $2pm - 6qn = \vec{0}$

Solving these, we get

$p = 1/(2m)$ and $q = 1/(3n)$

$\therefore \quad 0 < 1/(2m) \le 1$ and $0 < 1/(3n) \le 1$

$\Rightarrow \quad m \ge 1/2$ and $n \ge 1/3$

33. (1), (2), (4)

$$\left.\begin{aligned}\vec{V}_1 &= l\vec{a} + m\vec{b} + n\vec{c}\\ \vec{V}_2 &= n\vec{a} + l\vec{b} + m\vec{c}\\ \vec{V}_3 &= m\vec{a} + n\vec{b} + l\vec{c}\end{aligned}\right\}$$ when $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar.

Therefore,

$$[\vec{V}_1\,\vec{V}_2\,\vec{V}_3] = \begin{vmatrix} l & m & n \\ n & l & m \\ m & n & l \end{vmatrix} = 0$$

or $\quad (l + m + n)[(l - m)^2 + (m - n)^2 + (n - l)^2] = 0$

or $\quad l + m + n = 0$ (i)

Obviously, $lx^2 + mx + n = 0$ is satisfied by $x = 1$ due to (i).

$l^3 + m^3 + n^3 = 3lmn$

$\Rightarrow \quad (l + m + n)(l^2 + m^2 + n^2 - lm - mn - ln) = 0$, which is true

34. **(1), (2), (3)**

It is given that $\vec{\alpha}, \vec{b}$ and $\vec{\gamma}$ are coplanar vectors. Therefore,

$$[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = 0$$

$$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

or $3abc - a^3 - b^3 - c^3 = 0$

or $a^3 + b^3 + c^3 - 3abc = 0$

or $(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$

or $a + b + c = 0$

$[\because a^2 + b^2 + c^2 - ab - bc - ca \neq 0]$

$$\Rightarrow \vec{v}\cdot\vec{\alpha} = \vec{v}\cdot\vec{\beta} = \vec{v}\cdot\vec{\gamma} = 0$$

Hence, $\vec{v}$ is perpendicular to $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$

35. **(2), (4)**

For $\vec{A}, \vec{B}$ and $\vec{C}$ to form a left-handed system

$$[\vec{A}\vec{B}\vec{C}] < 0$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 1 & 5 \end{vmatrix} = 11\hat{i} - 6\hat{j} - \hat{k} \qquad \text{(i)}$$

(i) is satisfied by options (2) and (4).

36. **(1), (2), (3), (4)**

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$
$$= (yz + yx + zx)\{(y-z)\hat{i} + (z-x)\hat{j} + (x-y)\hat{k}\}$$

Clearly this vector is parallel to $(y-z)\hat{i} + (z-x)\hat{j} + (x-y)\hat{k}$.

It is orthogonal to $\hat{i} + \hat{j} + \hat{k}$ as $(y-z)(1) + (z-x)(1) + (x-y)(1) = 0$.

It is orthogonal to $(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ as $(y-z)(y+z) + (z-x)(z+x) + (x-y)(x+y)$
$= y^2 - z^2 + z^2 - x^2 + x^2 - y^2 = 0$

Also it is orthogonal to $x\hat{i} + y\hat{j} + z\hat{k}$

37. **(1), (3), (4)**

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$$

or $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}$

or $(\vec{a}\cdot\vec{b})\vec{c} - (\vec{c}\cdot\vec{b})\vec{a} = \vec{0}$

or $\vec{b}\times(\vec{c}\times\vec{a}) = \vec{0}$

or $(\vec{c}\times\vec{a})\times\vec{b} = \vec{0}$

or $\vec{b}\times(\vec{c}\times\vec{a}) = (\vec{c}\times\vec{a})\times\vec{b} = \vec{0}$

38. **(1), (4)**

$$\vec{a} = \hat{i} - \hat{j} + \hat{k},$$

$$\vec{b} = 2\hat{i} + \hat{j}$$

and $\vec{c} = 3\hat{j} - 2\hat{k}$.

Since $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & -2 \end{vmatrix} = 0$

Therefore, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors.

Further since $\vec{d}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$\therefore \quad \vec{d}\cdot\vec{a} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = 0$

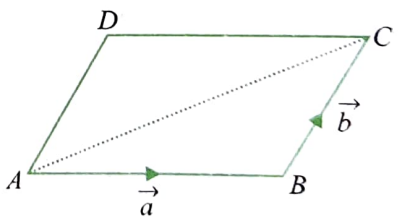
$\therefore \quad \vec{d}\cdot\vec{x} = \vec{d}\cdot\vec{y} = \vec{d}\cdot\vec{z} = 0$

$\therefore \quad \vec{d}\cdot\vec{r} = 0$

39. **(2), (3)**

$$\overrightarrow{AC} = \vec{a} + \vec{b}$$

$\therefore \quad |\overrightarrow{AC}| = |\vec{a} + \vec{b}|$



$\therefore$

$$|\overrightarrow{AC}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$$
$$= |3\vec{\alpha} - \vec{\beta}|^2 + |\vec{\alpha} + 3\vec{\beta}|^2 + 2(3\vec{\alpha} - \vec{\beta})\cdot(\vec{\alpha} + 3\vec{\beta})$$
$$= 16|\vec{\alpha}|^2 + 4|\vec{\beta}|^2 + 16\vec{\alpha}\cdot\vec{\beta}$$
$$= 80 + 16(2)(2)(1/2)$$
$$= 112$$

$\therefore \quad |\overrightarrow{AC}| = 4\sqrt{7}$

$$|\overrightarrow{BD}| = |\vec{a} - \vec{b}|$$

$$|\overrightarrow{BD}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$$
$$= |3\vec{\alpha} - \vec{\beta}|^2 + |\vec{\alpha} + 3\vec{\beta}|^2$$
$$- 2(3\vec{\alpha} - \vec{\beta})\cdot(\vec{\alpha} + 3\vec{\beta})$$
$$= 4|\vec{\alpha}|^2 + 4|\vec{\beta}|^2 - 16\vec{\alpha}\cdot\vec{\beta}$$
$$= 80 - 16(2)(2)(1/2)$$
$$= 48$$

$\therefore \quad |\overrightarrow{BD}| = 4\sqrt{3}$

Linked Comprehension Type

For Problems 1–3

1. (2), 2. (3), 3. (4)

Taking dot product of $\vec{u} + \vec{v} + \vec{w} = \vec{a}$ with $\vec{u}$, we have

$$1 + \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} = \vec{a}\cdot\vec{u} = \frac{3}{2} \text{ or } \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} = \frac{1}{2} \quad \text{(i)}$$

Similarly, taking dot product with $\vec{v}$, we have

$$\vec{u}\cdot\vec{v} + \vec{w}\cdot\vec{v} = \frac{3}{4} \quad \text{(ii)}$$

Also, $\vec{a}\cdot\vec{u} + \vec{a}\cdot\vec{v} + \vec{a}\cdot\vec{w} = \vec{a}\cdot\vec{a} = 4$

$$\Rightarrow \quad \vec{a}\cdot\vec{w} = 4 - \left(\frac{3}{2} + \frac{7}{4}\right) = \frac{3}{4}$$

Again, taking dot product with $\vec{w}$, we have

$$\vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w} = \frac{3}{4} - 1 = -\frac{1}{4} \quad \text{(iii)}$$

Adding (i), (ii) and (iii), we have

$$2(\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}) = 1$$

or $$\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w} = \frac{1}{2} \quad \text{(iv)}$$

Subtracting (i), (ii) and (iii) from (iv), we have

$$\vec{v}\cdot\vec{w} = 0, \quad \vec{u}\cdot\vec{w} = -\frac{1}{4} \text{ and } \vec{u}\cdot\vec{v} = \frac{3}{4}$$

Now, the equations $\vec{u}\times(\vec{v}\times\vec{w}) = \vec{b}$ and $(\vec{u}\times\vec{v})\times\vec{w} = \vec{c}$ can be written as

$$(\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w} = \vec{b}$$

and $$(\vec{u}\cdot\vec{w})\vec{v} - (\vec{v}\cdot\vec{w})\vec{u} = \vec{c}$$

$$\Rightarrow \quad -\frac{1}{4}\vec{v} - \frac{3}{4}\vec{w} = \vec{b}, \quad -\frac{1}{4}\vec{v} = \vec{c}, \text{ i.e., } \vec{v} = -4\vec{c}$$

$$\Rightarrow \quad \vec{c} - \frac{3}{4}\vec{w} = \vec{b} \Rightarrow \vec{w} = \frac{4}{3}(\vec{c} - \vec{b})$$

and $$\vec{u} = \vec{a} - \vec{v} - \vec{w} = \vec{a} + 4\vec{c} - \frac{4}{3}\vec{c} + \frac{4}{3}\vec{b}$$
$$= \vec{a} + \frac{4}{3}\vec{b} + \frac{8}{3}\vec{c}$$

For Problems 4–6

4. (4), 5. (3), 6. (2)

Given that $|\vec{x}| = |\vec{y}| = |\vec{z}| = \sqrt{2}$ and they are inclined at an angle of $60°$ with each other.

$\therefore \quad \vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{z} = \vec{z}\cdot\vec{x} = \sqrt{2}\cdot\sqrt{2}\cos 60° = 1$

$$\vec{x}\times(\vec{y}\times\vec{z}) = \vec{a}$$

or $$(\vec{x}\cdot\vec{z})\vec{y} - (\vec{x}\cdot\vec{y})\vec{z} = \vec{a} \text{ or } \vec{y} - \vec{z} = \vec{a} \quad \text{(i)}$$

Similarly, $\vec{y}\times(\vec{z}\times\vec{x}) = \vec{b} \Rightarrow \vec{z} - \vec{x} = \vec{b}$ (ii)

$$\vec{y} = \vec{a} + \vec{z}, \quad \vec{x} = \vec{z} - \vec{b}$$

[from (i) and (ii)] (iii)

Now, $\vec{x}\times\vec{y} = \vec{c}$

$\Rightarrow \quad (\vec{z} - \vec{b})\times(\vec{z} + \vec{a}) = \vec{c}$

or $$\vec{z}\times\vec{a} - \vec{b}\times\vec{z} - \vec{b}\times\vec{a} = \vec{c}$$

or $$\vec{z}\times(\vec{a} + \vec{b}) = \vec{c} + (\vec{b}\times\vec{a}) \quad \text{(iv)}$$

or $$(\vec{a} + \vec{b})\times\{\vec{z}\times(\vec{a} + \vec{b})\}$$
$$= (\vec{a} + \vec{b})\times\vec{c} + (\vec{a} + \vec{b})\times(\vec{b}\times\vec{a})$$

or $$(\vec{a} + \vec{b})^2\vec{z} - \{(\vec{a} + \vec{b})\cdot\vec{z}\}(\vec{a} + \vec{b})$$
$$= (\vec{a} + \vec{b})\times\vec{c} + |\vec{a}|^2\vec{b} - |\vec{b}|^2\vec{a}$$
$$+ (\vec{a}\cdot\vec{b})(\vec{b} - \vec{a}) \quad \text{(v)}$$

Now, (i) $\Rightarrow |\vec{a}|^2 = |\vec{y} - \vec{z}|^2 = 2 + 2 - 2 = 2$

Similarly, (ii) $\Rightarrow |\vec{b}|^2 = 2$

Also (i) and (ii) $\Rightarrow \vec{a} + \vec{b} = \vec{y} - \vec{x}$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = 2$ (vi)

Also $$(\vec{a} + \vec{b})\cdot\vec{z} = (\vec{y} - \vec{x})\cdot\vec{z} = \vec{y}\cdot\vec{z} - \vec{x}\cdot\vec{z}$$
$$= 1 - 1 = 0$$

and $$\vec{a}\cdot\vec{b} = (\vec{y} - \vec{z})\cdot(\vec{z} - \vec{x})$$
$$= \vec{y}\cdot\vec{z} - \vec{x}\cdot\vec{y} - |\vec{z}|^2 + \vec{x}\cdot\vec{z} = -1$$

Thus from (v), we have

$$2\vec{z} = (\vec{a} + \vec{b}) \times \vec{c} + 2(\vec{b} - \vec{a}) - (\vec{b} - \vec{a})$$

or $$\vec{z} = (1/2)[(\vec{a} + \vec{b}) \times \vec{c} + \vec{b} - \vec{a}]$$

$\therefore$ $$\vec{y} = \vec{a} + \vec{z} = (1/2)[(\vec{a} + \vec{b}) \times \vec{c} + \vec{b} + \vec{a}]$$

and $$\vec{x} = \vec{z} - \vec{b} = (1/2)[(\vec{a} + \vec{b}) \times \vec{c} - (\vec{a} + \vec{b})]$$

For Problems 7–9

7. (2), 8. (1), 9. (3)

Given

$$\vec{x} \times \vec{y} = \vec{a} \quad \text{(i)}$$

$$\vec{y} \times \vec{z} = \vec{b} \quad \text{(ii)}$$

$$\vec{x} \cdot \vec{b} = \gamma \quad \text{(iii)}$$

$$\vec{x} \cdot \vec{y} = 1 \quad \text{(iv)}$$

$$\vec{y} \cdot \vec{z} = 1 \quad \text{(v)}$$

From (ii), $\vec{x} \cdot (\vec{y} \times \vec{z}) = \vec{x} \cdot \vec{b} = \gamma \Rightarrow [\vec{x}\ \vec{y}\ \vec{z}] = \gamma$

From (i) and (ii), $(\vec{x} \times \vec{y}) \times (\vec{y} \times \vec{z}) = \vec{a} \times \vec{b}$

$\therefore$ $$[\vec{x}\ \vec{y}\ \vec{z}]\vec{y} - [\vec{y}\ \vec{y}\ \vec{z}]\vec{x} = \vec{a} \times \vec{b}$$

or $$\vec{y} = \frac{\vec{a} \times \vec{b}}{\gamma} \quad \text{(vi)}$$

Also from (i), we get $(\vec{x} \times \vec{y}) \times \vec{y} = \vec{a} \times \vec{y}$

or $$(\vec{x} \cdot \vec{y})\vec{y} - (\vec{y} \cdot \vec{y})\vec{x} = \vec{a} \times \vec{y}$$

or $$\vec{x} = (1/|\vec{y}|^2)(\vec{y} - \vec{a} \times \vec{y})$$

$$= \frac{\gamma^2}{|\vec{a} \times \vec{b}|^2}\left[\frac{\vec{a} \times \vec{b}}{\gamma} - \frac{\vec{a} \times (\vec{a} \times \vec{b})}{\gamma}\right]$$

$$= \frac{\gamma}{|\vec{a} \times \vec{b}|^2}[\vec{a} \times \vec{b} - \vec{a} \times (\vec{a} \times \vec{b})]$$

Also from (ii), $(\vec{y} \times \vec{z}) \times \vec{y} = \vec{b} \times \vec{y}$

or $$|\vec{y}|^2 \vec{z} - (\vec{z} \cdot \vec{y})\vec{y} = \vec{b} \times \vec{y}$$

or $$\vec{z} = \frac{1}{|\vec{y}|^2}[\vec{y} + \vec{b} \times \vec{y}]$$

$$= \frac{\gamma}{|\vec{a} \times \vec{b}|^2}[\vec{a} \times \vec{b} + \vec{b} \times (\vec{a} \times \vec{b})]$$

For Problems 10–12

10. (2), 11. (2), 12. (4)

$\vec{P} \times \vec{B} = \vec{A} - \vec{P}$ and $|\vec{A}| = |\vec{B}| = 1$ and $\vec{A} \cdot \vec{B} = 0$ is given

Now $$\vec{P} \times \vec{B} = \vec{A} - \vec{P} \quad \text{(i)}$$

$$(\vec{P} \times \vec{B}) \times \vec{B} = (\vec{A} - \vec{P}) \times \vec{B}$$

(taking cross product with $\vec{B}$ on both sides)

or $$(\vec{P} \cdot \vec{B})\vec{B} - (\vec{B} \cdot \vec{B})\vec{P} = \vec{A} \times \vec{B} - \vec{P} \times \vec{B}$$

or $$(\vec{P} \cdot \vec{B})\vec{B} - \vec{P} = \vec{A} \times \vec{B} - \vec{A} + \vec{P}$$

or $$2\vec{P} = \vec{A} - \vec{A} \times \vec{B} - (\vec{P} \cdot \vec{B})\vec{B}$$

or $$\vec{P} = \frac{\vec{A} - \vec{A} \times \vec{B} - (\vec{P} \cdot \vec{B})\vec{B}}{2} \quad \text{(ii)}$$

Taking dot product with $\vec{B}$ on both sides of (i), we get

$$\vec{P} \cdot \vec{B} = \vec{A} \cdot \vec{B} - \vec{P} \cdot \vec{B}$$

$\Rightarrow$ $$\vec{P} \cdot \vec{B} = 0 \quad \text{(iii)}$$

$\Rightarrow$ $$\vec{P} = \frac{\vec{A} + \vec{B} \times \vec{A}}{2}$$

Now

$$(\vec{P} \times \vec{B}) \times \vec{B} = (\vec{P} \cdot \vec{B})\vec{B} - (\vec{B} \cdot \vec{B})\vec{P} = -\vec{P}$$

$\vec{P}, \vec{A}, \vec{P} \times \vec{B}\ (= \vec{A} - \vec{P})$ are dependent

Also $$\vec{P} \cdot \vec{B} = 0$$

and $$|\vec{P}|^2 = \left|\frac{\vec{A} - \vec{A} \times \vec{B}}{2}\right|^2$$

$$= \frac{|\vec{A}|^2 + |\vec{A} \times \vec{B}|^2}{4}$$

$$= \frac{1+1}{4} = \frac{1}{2} \quad \text{or} \quad |\vec{P}| = \frac{1}{\sqrt{2}}$$

For Problems 13–15

13. (2) $$\vec{a}_1 = \left[(2\hat{i} + 3\hat{j} - 6\hat{k}) \cdot \frac{(2\hat{i} - 3\hat{j} + 6\hat{k})}{7}\right]\frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

$$= \frac{-41}{49}(2\hat{i} - 3\hat{j} + 6\hat{k})$$

$$\vec{a}_2 = \frac{-41}{49}\left((2\hat{i}-3\hat{j}+6\hat{k})\cdot\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}\right)$$

$$\times\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}$$

$$=\frac{-41}{(49)^2}(-4-9+36)(-2\hat{i}+3\hat{j}+6\hat{k})$$

$$=\frac{943}{49^2}(2\hat{i}-3\hat{j}-6\hat{k})$$

14. (1) $\vec{a}_1\cdot\vec{b}=\frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})\cdot(2\hat{i}-3\hat{j}+6\hat{k})=-41$

15. (3) $\vec{a}$, $\vec{a}_1$ and $\vec{b}$ are coplanar because $\vec{a}_1$ and $\vec{b}$ are collinear.

For Problems 16–18

16. (2), 17. (3), 18. (1)

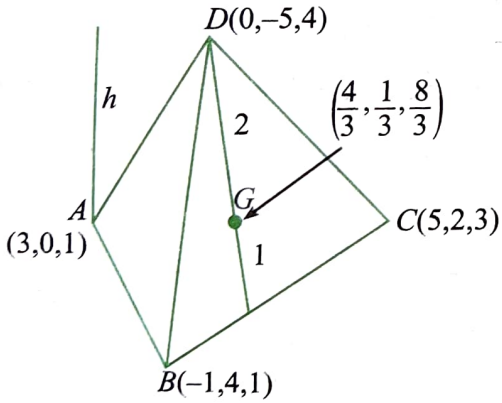
Point G is $\left(\frac{4}{3},\frac{1}{3},\frac{8}{3}\right)$. Therefore,

$$\left|\overrightarrow{AG}\right|^2=\left(\frac{5}{3}\right)^2+\frac{1}{9}+\left(\frac{5}{3}\right)^2=\frac{51}{9}$$

or $$\left|\overrightarrow{AG}\right|=\frac{\sqrt{51}}{3}$$

$$\overrightarrow{AB}=-4\hat{i}+4\hat{j}+0\hat{k}$$

$$\overrightarrow{AC}=2\hat{i}+2\hat{j}+2\hat{k}$$



$$\therefore\quad \overrightarrow{AB}\times\overrightarrow{AC}=-8\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1&0\\1&1&1\end{vmatrix}$$

$$=8(\hat{i}+\hat{j}-2\hat{k})$$

Area of $\Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=4\sqrt{6}$

$$\overrightarrow{AD}=-3\hat{i}-5\hat{j}+3\hat{k}$$

The length of the perpendicular from the vertex D on the opposite face

$$=|\text{ Projection of }\overrightarrow{AD}\text{ on }\overrightarrow{AB}\times\overrightarrow{AC}|$$

$$=\left|\frac{(-3\hat{i}-5\hat{j}+3\hat{k})(\hat{i}+\hat{j}-2\hat{k})}{\sqrt{6}}\right|$$

$$=\left|\frac{-3-5-6}{\sqrt{6}}\right|=\frac{14}{\sqrt{6}}$$

For Problems 19–21

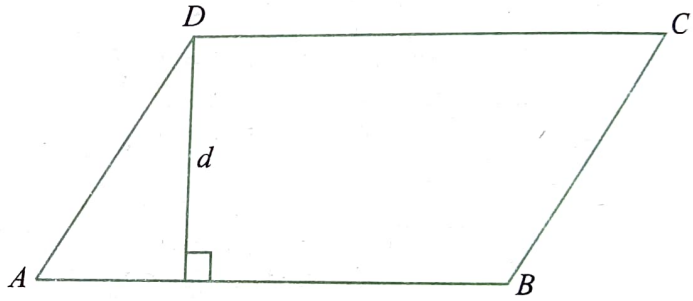
19. (3) Let point D be (a_1, a_2, a_3)

$a_1+1=3$ or $a_1=2$

$a_2+0=1$ or $a_2=1$

$a_3-1=7$ or $a_3=8$

$\therefore\quad D(2, 1, 8)$



$$\vec{d}=\left|\frac{(\overrightarrow{AB})\times(\overrightarrow{AD})}{|\overrightarrow{AB}|}\right|$$

$$\overrightarrow{AB}=-\hat{i}+\hat{j}-5\hat{k}$$

$$\overrightarrow{AD}=0\hat{i}+2\hat{j}+4\hat{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\-1&1&-5\\0&2&4\end{vmatrix}$$

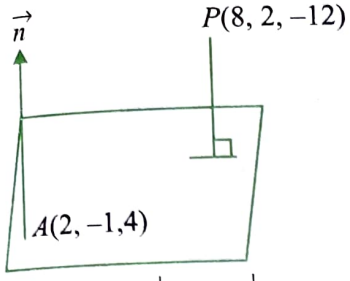
$$=14\hat{i}+4\hat{j}-2\hat{k}$$

$$=2(7\hat{i}+2\hat{j}-\hat{k})$$

$$\Rightarrow\quad d=2\sqrt{2}$$

20. (2) $\vec{n}=7\hat{i}+2\hat{j}-\hat{k}$ is normal to the plane $P\equiv(8, 2, -12)$.

$$|\overrightarrow{AP}|=6\hat{i}+3\hat{j}-16\hat{k}$$



$$\therefore \text{ Distance, } d = \left|\frac{\overrightarrow{AP}\cdot\vec{n}}{|\vec{n}|}\right|$$

$$= \left|\frac{42+6+16}{\sqrt{49+4+1}}\right|$$

$$= \frac{64}{\sqrt{54}}$$

$$= \frac{64}{3\sqrt{6}} = \frac{64\sqrt{6}}{18} = \frac{32\sqrt{6}}{9}$$

21. (4) Vector normal to the plane

$$\overrightarrow{AD}\times\overrightarrow{AB} = +(7\hat{i}+2\hat{j}-\hat{k})$$

Projection on $xy = 2$

Projection on $yz = 14$

Projection on $zx = 4$

For Problems 22–24

22. (4), 23. (3), 24. (3)

Let $\vec{r} = x\hat{i}+y\hat{j}$

$x^2+y^2+8x-10y+40=0$, which is a circle

centre $C(-4, 5)$, radius $r = 1$

$p_1 = \max\{(x+2)^2+(y-3)^2\}$

$p_2 = \min\{(x+2)^2+(y-3)^2\}$

Let P be $(-2, 3)$. Then

$CP = \sqrt{2}, r = 1$

$p_2 = (2\sqrt{2}-1)^2$

$p_1 = (2\sqrt{2}+1)^2$

$p_1+p_2 = 18$

$$\text{Slope} = AB = \left(\frac{dy}{dx}\right)_{(2,2)} = -2$$

Equation of AB, $2x+y=6$

$$\overrightarrow{OA} = 2\hat{i}+2\hat{j},\quad \overrightarrow{OB} = 3\hat{i}$$

$$\overrightarrow{AB} = \hat{i}-2\hat{j}$$

$$\overrightarrow{AB}\cdot\overrightarrow{OB} = (\hat{i}-2\hat{j})(3\hat{i}) = 3$$

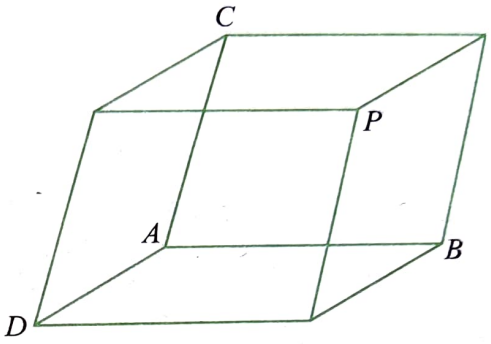
For Problems 25–27

25. (1), 26. (2), 27. (3)

$$\vec{a} = \overrightarrow{AP} = \overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD} \qquad \text{(i)}$$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \vec{b}$$

$$\overrightarrow{AD}\times\overrightarrow{AB} = \vec{c}$$



$$\overrightarrow{AB}\cdot\frac{\vec{a}}{|\vec{a}|} = \frac{|\vec{a}|}{3} \Rightarrow \overrightarrow{AB}\cdot\vec{a} = \frac{|\vec{a}|^2}{3}$$

$$\overrightarrow{AC}\cdot\frac{\vec{a}}{|\vec{a}|} = \frac{|\vec{a}|}{3} \Rightarrow \overrightarrow{AC}\cdot\vec{a} = \frac{|\vec{a}|^2}{3}$$

$$\therefore \quad (\overrightarrow{AB}\times\overrightarrow{AC})\times\vec{a} = \vec{b}\times\vec{a}$$

$$\therefore \quad \frac{|\vec{a}|^2}{3}\overrightarrow{AC}-\frac{|\vec{a}|^2}{3}\overrightarrow{AB} = \vec{b}\times\vec{a}$$

$$\therefore \quad \overrightarrow{AC}-\overrightarrow{AB} = 3\frac{\vec{b}\times\vec{a}}{|\vec{a}|^2} \qquad \text{(ii)}$$

$$|\vec{a}|^2 = \overrightarrow{AB}\cdot\vec{a}+\overrightarrow{AC}\cdot\vec{a}+\overrightarrow{AD}\cdot\vec{a}$$

$$\therefore \quad \frac{|\vec{a}|^2}{3} = \overrightarrow{AD}\cdot\vec{a}$$

$$(\overrightarrow{AD}\times\overrightarrow{AB})\times\vec{a} = \vec{c}\times\vec{a}$$

$$\overrightarrow{AB}-\overrightarrow{AD} = 3\frac{\vec{c}\times\vec{a}}{|\vec{a}|^2} \qquad \text{(iii)}$$

from (i), (ii) and (iii), we get

$$\overrightarrow{AB} = \frac{1}{3}\vec{a}+\frac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2}$$

Now from (ii) and (iii), we get $\overrightarrow{AC}$ and $\overrightarrow{AD}$ as

$$\overrightarrow{AC} = \frac{1}{3}\vec{a}+\frac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2}+\frac{3(\vec{b}\times\vec{a})}{|\vec{a}|^2}$$

$$\overrightarrow{AD} = \frac{1}{3}\vec{a}+\frac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2}-\frac{3(\vec{c}\times\vec{a})}{|\vec{a}|^2}$$

Matrix Match Type

1. **a → p, q, r, s; b → p, q; c → p, r; d → r**

a. Given equations are consistent if

$$(\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}-\hat{k})$$

$$=(\hat{i}+2\hat{j})+\mu(-\hat{i}+\hat{j}+a\hat{k})$$

$\Rightarrow \quad 1+\lambda=1-\mu,\ 1+2\lambda=2+\mu,\ -\lambda=a\mu$

$\Rightarrow \quad \lambda=1/3$ and $\mu=-1/3$

$\Rightarrow \quad a=1$

b. $\vec{a}=\lambda\hat{i}-3\hat{j}-\hat{k}$

$$\vec{b}=2\lambda\hat{i}+\lambda\hat{j}-\hat{k}$$

Angle between $\vec{a}$ and $\vec{b}$ is acute. Therefore,

$$\vec{a}\cdot\vec{b}>0$$

$\Rightarrow \quad 2\lambda^2-3\lambda+1>0$

or $\quad (2\lambda-1)(\lambda-1)>0$

or $\quad \lambda\in\left(-\infty,\frac{1}{2}\right)\cup(1,\infty)$

Also $\vec{b}$ makes an obtuse angle with the axes. Therefore,

$$\vec{b}\cdot\hat{i}<0\Rightarrow\lambda<0$$

$$\vec{b}\cdot\hat{j}<0\Rightarrow\lambda<0 \qquad \text{(ii)}$$

Combining these two, we get $\lambda=-4,-2$

c. If vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}+2\hat{j}+(1+a)\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar, then

$$\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & 1+a\\ 3 & a & 5\end{vmatrix}=0$$

or $\quad a^2+2a-8=0$

or $\quad (a+4)(a-2)=0$

or $\quad a=-4,2$

d. $\vec{A}=2\hat{i}+\lambda\hat{j}+3\hat{k}$

$B=2\hat{i}+\lambda\hat{j}+\hat{k}$

$C=3\hat{i}+\hat{j}+0\cdot\hat{k}$

$\vec{A}+\lambda\vec{B}=2(1+\lambda)\hat{i}+(\lambda+\lambda^2)\hat{j}+(3+\lambda)\hat{k}$

Now $(\vec{A}+\lambda\vec{B})\perp\vec{C}$.

Therefore, $(\vec{A}+\lambda\vec{B})\cdot\vec{C}=0$

or $\quad 6(1+\lambda)+(\lambda+\lambda^2)+0=0$

or $\quad \lambda^2+7\lambda+6=0$

or $\quad (\lambda+6)(\lambda+1)=0$

or $\quad \lambda=-6,=-1$

$\Rightarrow \quad |2\lambda|=12,2$

2. **a → r; b → p; c → s; d → q**

a. If $\vec{a},\ \vec{b}$ and $\vec{c}$ are mutually perpendicular, then

$$[\vec{a}\times\vec{b}\quad\vec{b}\times\vec{c}\quad\vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]^2$$

$$=(|\vec{a}||\vec{b}||\vec{c}|)^2=16$$

b. Given $\vec{a}$ and $\vec{b}$ are two unit vectors, i.e., $|\vec{a}|=|\vec{b}|=1$ and angle between them is $\pi/3$.

$$\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\Rightarrow\sin\frac{\pi}{3}=|\vec{a}\times\vec{b}|$$

$$\Rightarrow\frac{\sqrt{3}}{2}=|\vec{a}\times\vec{b}|$$

Now

$$[\vec{a}\quad\vec{b}+\vec{a}\times\vec{b}\quad\vec{b}]=[\vec{a}\ \vec{b}\ \vec{b}]+[\vec{a}\quad\vec{a}\times\vec{b}\quad\vec{b}]$$

$$=0+[\vec{a}\quad\vec{a}\times\vec{b}\quad\vec{b}]$$

$$=(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{a})$$

$$=-(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$$

$$=-|\vec{a}\times\vec{b}|^2$$

$$=-\frac{3}{4}$$

c. If $\vec{b}$ and $\vec{c}$ are orthogonal $\vec{b}\cdot\vec{c}=0$.

Also, it is given that $\vec{b}\times\vec{c}=\vec{a}$. Now

$$[\vec{a}+\vec{b}+\vec{c}\quad\vec{a}+\vec{b}\quad\vec{b}+\vec{c}]$$

$$=[\vec{a}\quad\vec{a}+\vec{b}\quad\vec{b}+\vec{c}]+[\vec{b}+\vec{c}\quad\vec{a}+\vec{b}\quad\vec{b}+\vec{c}]$$

$$=[\vec{a}\ \vec{b}\ \vec{c}]$$

$$=\vec{a}\cdot(\vec{b}\times\vec{c})$$

$$=\vec{a}\cdot\vec{a}=|\vec{a}|^2=1 \quad \text{(because } \vec{a} \text{ is a unit vector)}$$

d. $[\vec{x}\ \vec{y}\ \vec{a}] = 0$

Therefore $\vec{x}$, $\vec{y}$ and $\vec{a}$ are coplanar. (i)

$[\vec{x}\ \vec{y}\ \vec{b}] = 0$

Therefore, $\vec{x}$, $\vec{y}$ and $\vec{b}$ are coplanar. (ii)

Also, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Therefore, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar (iii)

From (i), (ii) and (iii),

$\vec{x}$, $\vec{y}$ and $\vec{c}$ are coplanar. Therefore,

$[\vec{x}\ \vec{y}\ \vec{c}] = 0$

3. a → q; b → s; c → p; d → r

a. $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{6}$

$\Rightarrow \vec{a}^2+\vec{b}^2+\vec{c}^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a} = 6$

$\therefore \quad |\vec{a}| = 1$

b. $\vec{a}$ is perpendicular to $\vec{b}+\vec{c}$

$\Rightarrow \quad \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} = 0$ (i)

$\vec{b}$ is perpendicular to $\vec{a}+\vec{c}$

$\Rightarrow \quad \vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{c} = 0$ (ii)

$\vec{c}$ is perpendicular to $\vec{a}+\vec{b}$

$\Rightarrow \quad \vec{c}\cdot\vec{a}+\vec{a}\cdot\vec{c}\cdot\vec{b} = 0$ (iii)

From (i), (ii) and (iii), we get

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}| = 7$

c. $(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d}) = 21$

d. We know that

$$[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$$

and $$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{vmatrix}$$

$$= 32$$

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = 4\sqrt{2}$

4. a → s; b → r; c → q; d → p

a. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ -1 & -2 & -1 \end{vmatrix} = 3\hat{i}-3\hat{j}+3\hat{k}$

Hence, the area of the triangle is $\frac{3\sqrt{3}}{2}$.

b. The area of the parallelogram is $3\sqrt{3}$.

c. The area of a parallelogram whose diagonals are $2\vec{a}$ and $4\vec{b}$ is $\frac{1}{2}|2\vec{a}\times 4\vec{b}| = 12\sqrt{3}$.

d. Volume of the parallelepiped

$= |(\vec{a}\times\vec{b})\cdot\vec{c}| = \sqrt{9+36+9} = 3\sqrt{6}$

5. a → p, r; b → q; c → p, q, s; d → p

a. Vector $-3\hat{i}+3\hat{j}+4\hat{k}$ and $\hat{i}+\hat{j}$ are coplanar with $\vec{a}$ and $\vec{b}$.

b. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ -2 & 1 & 2 \end{vmatrix}$

$= 2\hat{i}-2\hat{j}+3\hat{k}$

c. If $\vec{c}$ is equally inclined to $\vec{a}$ and $\vec{b}$, then we must have $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$,

which is true for vectors in options p, q, s

d. Vector is forming a triangle with $\vec{a}$ and $\vec{b}$. Then

$\vec{c} = \vec{a}+\vec{b} = -3\hat{i}+3\hat{j}+4\hat{k}$

6. a → q; b → s; c → p; d → r

a. $|\vec{a}+\vec{b}| = |\vec{a}+2\vec{b}|$

$a^2+b^2+2\vec{a}\cdot\vec{b} = a^2+4b^2+4\vec{a}\cdot\vec{b}$

or $\quad 2\vec{a}\cdot\vec{b} = -3b^2 < 0$

Hence, angle between $\vec{a}$ and $\vec{b}$ is obtuse.

b. $|\vec{a}+\vec{b}| = |\vec{a}-2\vec{b}|$

or $\quad a^2+b^2+2\vec{a}\cdot\vec{b} = a^2+4b^2-4\vec{a}\cdot\vec{b}$

or $\quad 6\vec{a}\cdot\vec{b} = 3b^2$

Hence, angle between $\vec{a}$ and $\vec{b}$ is acute.

c. $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$\Rightarrow \quad \vec{a}\cdot\vec{b}$

Hence, $\vec{a}$ is perpendicular to $\vec{b}$.

d. $\vec{c}\times(\vec{a}\times\vec{b})$ lies in the plane of vectors $\vec{a}$ and $\vec{b}$.

A vector perpendicular to this plane is parallel to $\vec{a}\times\vec{b}$

Hence, angle is 0°.

7. **a → r; b → s; c → q; d → p**

$$[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = 36$$

or $\quad [\vec{a}\ \vec{b}\ \vec{c}] = 6$

$\Rightarrow$ Volume of tetrahedron formed by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is $\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]] = 1.$

$$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}] = 12$$

$\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$ and $\vec{c}-\vec{a}$ are coplanar

$\Rightarrow \quad [\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

Numerical Value Type

1. (5) Let angle between $\vec{a}$ and $\vec{b}$ be θ.

We have $|\vec{a}| = |\vec{b}| = 1$

Now $|\vec{a}+\vec{b}| = 2\cos\frac{\theta}{2}$ and $|\vec{a}-\vec{b}| = 2\sin\frac{\theta}{2}$

Consider $F(\theta) = \frac{3}{2}\left(2\cos\frac{\theta}{2}\right)+2\left(2\sin\frac{\theta}{2}\right)$

$\therefore \quad F(\theta) = 3\cos\frac{\theta}{2} + 4\sin\frac{\theta}{2},\ \theta\in[0,\pi]$

2. (1) Since angle between $\vec{u}$ and $\hat{i}$ is 60°, we have

$$\vec{u}\cdot i = |\vec{u}||\hat{i}|\cos 60^\circ = \frac{|\vec{u}|}{2}$$

Given that $|\vec{u}-\hat{i}|, |\vec{u}|, |\vec{u}-2\hat{i}|$ are in G.P., so $|\vec{u}-\hat{i}|^2 = |\vec{u}||\vec{u}-2\hat{i}|$

Squaring both sides,

$$[|\vec{u}|^2+|\hat{i}|^2-2\vec{u}\cdot\hat{i}]^2 = |\vec{u}|^2[|\vec{u}|^2+4|\hat{i}|^2-4\vec{u}\cdot\hat{i}]$$

$$[|\vec{u}|^2+1-\frac{2|\vec{u}|}{2}]^2 = |\vec{u}|^2[|\vec{u}|^2+4-4\frac{|\vec{u}|}{2}]$$

or $\quad |\vec{u}|^2+2|\vec{u}|-1=0 \Rightarrow |\vec{u}| = -\frac{2\pm 2\sqrt{2}}{2}$

or $\quad |\vec{u}| = \sqrt{2}-1$

3. (2) $\overrightarrow{AB} = 2\hat{i}+\hat{j}+\hat{k}$, $\overrightarrow{AC} = (t+1)\hat{i}+0\hat{j}-\hat{k}$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ t+1 & 0 & -1 \end{vmatrix}$$

$$= -\hat{i}+(t+3)\hat{j}-(t+1)\hat{k}$$

$$= \sqrt{1+(t+3)^2+(t+1)^2}$$

$$= \sqrt{2t^2+8t+11}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$

$$= \frac{1}{2}\sqrt{2t^2+8t+1}$$

Let $\quad f(t) = \Delta^2 = \frac{1}{4}(2t^2+8t+1)$

$f'(t) = 0 \Rightarrow t = -2$

At $t=-2$, $f''(t) > 0$

So Δ is minimum at $t=-2$

4. (7) $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

$\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

$$\text{L.H.S.} = [3\vec{a}+\vec{b}\ \ 3\vec{b}+\vec{c}\ \ 3\vec{c}+\vec{a}]$$

$$= [3\vec{a}\ 3\vec{b}\ 3\vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]$$

$$= 3^3[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{c}]$$

$$= 28[\vec{a}\ \vec{b}\ \vec{c}]$$

5. (4) $\vec{a} = \alpha\hat{i}+2\hat{j}-3\hat{k}$, $\vec{b} = \hat{i}+2\alpha\hat{j}-2\hat{k}$,

$\vec{c} = 2\hat{i}-\alpha\hat{j}+\hat{k}$ $\{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\}\times(\vec{c}\times\vec{a}) = \vec{0}$

or $\quad \{[\vec{a}\ \vec{b}\ \vec{c}]\vec{b}-[\vec{a}\ \vec{b}\ \vec{b}]\vec{c}\}\times(\vec{c}\times\vec{a}) = \vec{0}$

or $\quad [\vec{a}\ \vec{b}\ \vec{c}]\vec{b}\times(\vec{c}\times\vec{a}) = \vec{0}$

or $\quad [\vec{a}\ \vec{b}\ \vec{c}]\ ((\vec{a}\cdot\vec{b})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}) = \vec{0}$

or $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ ($\because \vec{a}$ and $\vec{c}$ are not collinear)

$$\Rightarrow \begin{vmatrix} \alpha & 2 & -3 \\ 1 & 2\alpha & -2 \\ 2 & -\alpha & 1 \end{vmatrix}$$

or $\alpha(2\alpha - 2\alpha) - 2(1 + 4) - 3(-\alpha - 4\alpha) = 0$

or $10 - 15\alpha = 0$

$\therefore \quad \alpha = 2/3$

(9) Since $\vec{x}$ and $\vec{y}$ are non-collinear vectors, therefore $\vec{x}, \vec{y}$ and $\vec{x} \times \vec{y}$ are non-coplanar vectors.

$[(a-2)\alpha^2 + (b-3)\alpha + c] + [(a-2)\beta^2 + (b-3)\beta\beta + c]\vec{y} + [(a-2)\gamma^2 + (b-3)\gamma + c](\vec{x} \times \vec{y}) = 0$

Coefficient of each vector $\vec{x}$, $\vec{y}$ and $\vec{x} \times \vec{y}$ is zero.

$$(a-2)\alpha^2 + (b-3)\alpha + c = 0$$
$$(a-2)\beta^2 + (b-3)\beta + c = 0$$
$$(a-2)\gamma^2 + (b-3)\beta + c = 0$$

The above three equations will satisfy if the coefficients of α, β and γ are zero because α, β and γ are three distinct real numbers

$a - 2 = 0$ or $a = 2$,

$b - 3 = 0$ or $b = 3$ and $c = 0$

$\therefore \quad a^2 + b^2 + c^2 = 2^2 + 3^2 + 0^2 = 4 + 9 = 13$

7. (1) Given, $\vec{u} \times \vec{v} + \vec{u} = \vec{w}$ and $\vec{w} \times \vec{u} = \vec{v}$

$$\Rightarrow (\vec{u} \times \vec{v} + \vec{u}) \times \vec{u} = \vec{v}$$
$$\Rightarrow (\vec{u} \times \vec{v}) \times \vec{u} = \vec{v}$$
$$\Rightarrow \vec{v} - (\vec{u} \cdot \vec{v})\vec{u} = \vec{v}$$
$$\Rightarrow (\vec{u} \cdot \vec{v})\vec{u} = 0 \Rightarrow (\vec{u}\ \vec{v}) = 0$$

Now, $[\vec{u}\ \vec{v}\ \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$

$$= \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v} + \vec{u}))$$
$$= \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v}) + \vec{v} \times \vec{u})$$
$$= \vec{u}(v^2 \vec{u} - (\vec{u} \cdot \vec{v})\vec{v} + \vec{v} \times \vec{u}) = \vec{v}^2 \vec{u}^2 = 1$$

8. (7) Let the vertices be A, B, C, D and O be the origin.

$$\therefore \quad \vec{OA} = \hat{i} - 6\hat{j} + 10\hat{k}, \quad \vec{OB} = \hat{i} - 3\hat{j} + 7\hat{k},$$
$$\vec{OC} = -5\hat{i} - \hat{j} + \lambda\hat{k}, \quad \vec{OD} = 7\hat{i} - 4\hat{j} + 7\hat{k}$$
$$\therefore \quad \vec{AB} = \vec{OB} - \vec{OA} = -2\hat{i} + 3\hat{j} - 3\hat{k}$$
$$\vec{AC} = \vec{OC} - \vec{OA} = 4\hat{i} + 5\hat{j} + (\lambda - 10)\hat{k}$$
$$\vec{AD} = \vec{OD} - \vec{OA} = 6\hat{i} + 2\hat{j} - 3\hat{k}$$

Volume of tetrahedron

$$= \frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}] = \frac{1}{6}\begin{vmatrix} -2 & 3 & -3 \\ 4 & 5 & \lambda - 10 \\ 6 & 2 & -3 \end{vmatrix}$$

$$= \frac{1}{6}\{-2(-15 - 2\lambda + 20) - 3(-12 - 6\lambda + 60) - 3(8 - 30)\}$$

$$= \frac{1}{6}\{4\lambda - 10 - 144 + 18\lambda + 66\}$$

$$= \frac{1}{6}(22\lambda - 88) = 11 \qquad \text{(given)}$$

or $2\lambda - 8 = 6$

or $\lambda = 7$

9. (6) Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k}$; $\vec{v} = 2\hat{i} + \hat{j} + 4\hat{k}$; $\vec{w} = \hat{i} + 3\hat{j} + 3\hat{k}$

$(\vec{u} \cdot \vec{R} - 15)\hat{i} + (\vec{v} \cdot \vec{R} - 30)\hat{j} + (\vec{w} \cdot \vec{R} - 25)\hat{k} = \vec{0}$ (given)

So $\vec{u} \cdot \vec{R} = 15 \Rightarrow x - 2y + 3z = 15$ (i)

$\vec{v} \cdot \vec{R} = 30 \Rightarrow 2x + y + 4z = 30$ (ii)

$\vec{w} \cdot \vec{R} = 25 \Rightarrow x + 3y + 3z = 25$ (iii)

Solving, we get

$x = 4$

$y = 2$

$z = 5$

10. (6) $2\vec{V} + \vec{V} \times (\hat{i} + 2\hat{j}) = (2\hat{i} + \hat{k})$ (i)

or $2\vec{V} \cdot (\hat{i} + 2\hat{j}) + 0 = (2\hat{i} + \hat{k}) \cdot (\hat{i} + 2\hat{j})$

or $2\vec{V} \cdot (\hat{i} + 2\hat{j}) = 2$

or $|\vec{V} \cdot (\hat{i} + 2\hat{j})|^2 = 1$

or $|\vec{V}|^2 \cdot |\hat{i} + 2\hat{j}|^2 \cos^2\theta = 1$

(θ is the angle between $\vec{V}$ and $\hat{i} + 2\hat{j}$)

or $|\vec{V}|^2\ 5(1 - \sin^2\theta) = 1$

or $|\vec{V}|^2 5\sin^2\theta = 5|\vec{V}|^2 - 1$ (ii)

From Eq. (i), we have

$$|2\vec{V} + \vec{V} \times (\hat{i} + 2\hat{j})|^2 = |2\hat{i} + \hat{k}|^2$$

or $4|\vec{V}|^2 + |\vec{V} \times (\hat{i} + 2\hat{j})|^2 = 5$

or $4|\vec{V}|^2 + |\vec{V}|^2 \cdot |\hat{i} + 2\hat{j}|^2 \sin^2\theta = 5$

or $4|\vec{V}|^2 + 5|\vec{V}|^2 \sin^2\theta = 5$

or $4|\vec{V}|^2 + 5|\vec{V}|^2 - 1 = 5$

or $9|\vec{V}|^2 = 6$

or $3|\vec{V}| = \sqrt{6}$

$= \sqrt{6} = \sqrt{m}$

$\therefore \quad m = 6$

11.(1) $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\vec{a}\cdot\vec{c} = 0 \Rightarrow \vec{a} \perp \vec{c}$

$\Rightarrow \quad \vec{a} \perp \vec{b} - \vec{c}$

$|\vec{a}\times\vec{b} - \vec{a}\times\vec{c}| = |\vec{a}\times(\vec{b}-\vec{c})|$

$= |\vec{a}||\vec{b}-\vec{c}| = |\vec{b}-\vec{c}|$

Now $|\vec{b}-\vec{c}|^2 = |\vec{b}|^2 + |\vec{c}|^2 - 2|\vec{b}||\vec{c}|\cos\frac{\pi}{3}$

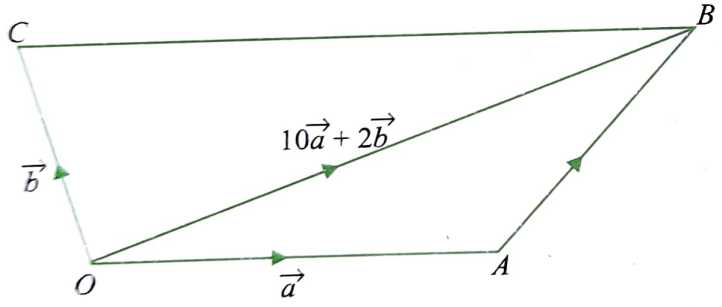
$= 2 - 2x \times \frac{1}{2} = 1$

$|\vec{b}-\vec{c}| = 1$

12. (6) Here $\vec{OA} = \vec{a}, \vec{OB} = 10\vec{a} + 2\vec{b}, \vec{OC} = \vec{b}$

q = Area of parallelogram with OA and OC as adjacent sides.

$\therefore \quad q = |\vec{a}\times\vec{b}|$ (i)



p = Area of quadrilateral $OABC$

= Area of ΔOAB + Area of ΔOBC

$= \frac{1}{2}|\vec{a}\times(10\vec{a}+2\vec{b})| + \frac{1}{2}|(10\vec{a}+2\vec{b})\times\vec{b}|$

$= |\vec{a}\times\vec{b}| + 5|\vec{a}\times\vec{b}|$

$\therefore \quad p = 6|\vec{a}\times\vec{b}|$

or $p = 6q$ [From Eq. (i)]

$\therefore \quad k = 6$

13. (9) Here $\vec{F} = 3\hat{i} - \hat{j} - 2\hat{k}$

$\vec{AB}$ = P.V. of B − P.V. of A

$= (-\hat{i} - \hat{j} - 2\hat{k}) - (-3\hat{i} - 4\hat{j} + \hat{k})$

$= 2\hat{i} + 3\hat{j} - 3\hat{k}$

Let $\vec{s} = \vec{AB}$ be the displacement vector

Now work done $= \vec{F}\cdot\vec{s}$

$= (3\hat{i} - \hat{j} - 2\hat{k})\cdot(2\hat{i} + 3\hat{j} - 3\hat{k})$

$= 6 - 3 + 6 = 9$

Archives

JEE Main

Single Correct Answer Type

1. (1) $(3p^2 - pq + 2q^2)[\vec{u}\ \vec{v}\ \vec{w}] = 0$

But $[\vec{u}\ \vec{v}\ \vec{w}] \neq 0$. So

$3p^2 - pq + 2q^2 = 0$

$\Rightarrow \quad 2p^2 + p^2 - pq + \left(\frac{q}{2}\right)^2 + \frac{7q^2}{4} = 0$

$\Rightarrow \quad 2p^2 + \left(p - \frac{q}{2}\right)^2 + \frac{7}{4}q^2 = 0$

$\Rightarrow \quad p = 0, q = 0, p = \frac{q}{2}$

This is possible only when $p = 0$, $q = 0$, i.e., exactly one value of (p, q).

2. (2) $\vec{c} = \vec{b}\times\vec{a}$

$\Rightarrow \quad \vec{b} \perp \vec{c}$

$\Rightarrow \quad (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\cdot(\hat{i} - \hat{j} - \hat{k}) = 0$

$\Rightarrow \quad b_1 - b_2 - b_3 = 0$

and $\vec{a}\cdot\vec{b} = 3$

$\Rightarrow \quad b_2 - b_3 = 3$

$\therefore \quad b_1 = b_2 + b_3 = 3 + 2b_3$

$\therefore \quad \vec{b} = (3 + 2b_3)\hat{i} + (3 + b_3)\hat{j} + b_3\hat{k}$

For $b_3 = -2$

$\vec{b} = -\hat{i} + \hat{j} - 2\hat{k}$

3. (2) $\vec{a}\cdot\vec{b} = 2 - 4 + 2 = 0$

Also, we must have $\vec{a}\cdot\vec{c} = 0$ and $\vec{b}\cdot\vec{c} = 0$

So, $\lambda - 1 + 2\mu = 0$ and $2\lambda + 4 + \mu = 0$

Solving we get $\lambda = -3$ and $\mu = 2$.

4. (2) We have

$\vec{a} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$

$$\vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$$

$\therefore \ \vec{a}\cdot\vec{b} = 0$

$$(2\vec{a} - \vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a} + 2\vec{b})]$$

$$= (2\vec{a} - \vec{b})\cdot[a^2\vec{b} - (\vec{a}\cdot\vec{b})\vec{b} + 2(\vec{a}\cdot\vec{b})\vec{b} - 2b^2\vec{a}]$$

$$= (2\vec{a} - \vec{b})\cdot(\vec{b} - 2\vec{a})$$

$$= -(2\vec{a} - \vec{b})^2$$

$$= -[4|\vec{a}|^2 + |\vec{b}|^2 - 4\vec{a}\cdot\vec{b}]$$

$$= -[4 + 1] = -5$$

5. (1) We have

$$\vec{b}\times\vec{c} = \vec{b}\times\vec{d}$$

$$\Rightarrow \ \vec{a}\times(\vec{b}\times\vec{c}) = \vec{a}\times(\vec{b}\times\vec{d})$$

$$\Rightarrow \ (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{d})\vec{b} - (\vec{a}\cdot\vec{b})\vec{d}$$

$$(\because \vec{a}\cdot\vec{d} = 0)$$

$$\Rightarrow \ (\vec{a}\cdot\vec{b})\vec{d} = -(\vec{a}\cdot\vec{c})\vec{b} + (\vec{a}\cdot\vec{b})\vec{c}$$

$$\Rightarrow \ \vec{d} = \vec{c} - \left(\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{b}$$

6. (3) $\vec{c} = \hat{a} + 2\hat{b}$

$$\vec{d} = 5\hat{a} - 4\hat{b}$$

$$\vec{c}\cdot\vec{d} = 0$$

$$\Rightarrow \ (\hat{a} + 2\hat{b})\cdot(5\hat{a} - 4\hat{b}) = 0$$

$$\Rightarrow \ 5 + 6\hat{a}\cdot\hat{b} - 8 = 0$$

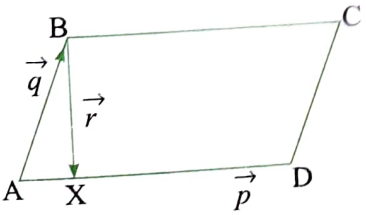
$$\Rightarrow \ \hat{a}\cdot\hat{b} = \frac{1}{2}$$

$$\Rightarrow \ a\cdot b\cos\theta = \frac{1}{2}$$

$$\Rightarrow \ \cos\theta = \frac{1}{2}$$

$$\Rightarrow \ \theta = \frac{\pi}{3}$$

7. (2)



$$\overrightarrow{AX} = \frac{\vec{p}\cdot\vec{q}}{|\vec{p}|}\frac{\vec{p}}{|\vec{p}|} = \frac{\vec{p}\cdot\vec{q}}{|\vec{p}|^2}\vec{p}$$

$$\overrightarrow{BX} = \overrightarrow{BA} + \overrightarrow{AX}$$

$$= -\vec{q} + \frac{\vec{p}\cdot\vec{q}}{|\vec{p}|^2}\vec{p}$$

8. (4) $[(\vec{a}\times\vec{b})(\vec{b}\times\vec{c})(\vec{c}\times\vec{a})] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

$\therefore \quad \lambda = 1$

9. (1) $(\vec{a}\times\vec{b})\times\vec{c} = \frac{1}{3}|b||c|\vec{a}$

or $\quad -(\vec{c}\cdot\vec{b})\vec{a} + (\vec{c}\cdot\vec{a})\vec{b} = \frac{1}{3}|b||c|\vec{a}$

or $\quad \left(\frac{1}{3}|b||c| + (\vec{c}\cdot\vec{b})\right)\vec{a} = (\vec{c}\cdot\vec{a})\vec{b}$

Since $\vec{a}$ and $\vec{b}$ are not collinear,

$$\vec{c}\cdot\vec{b} + \frac{1}{3}|b||c| = 0 \text{ and } \vec{c}\cdot\vec{a} = 0$$

$\therefore \quad \cos\theta + \frac{1}{3} = 0$

$\Rightarrow \quad \cos\theta = -\frac{1}{3}$ or $\sin\theta = \frac{2\sqrt{2}}{3}$.

10. (3) $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$

$$\Rightarrow \ (\vec{a}\,.\,\vec{c})\,\vec{b} - (\vec{a}\,.\,\vec{b})\,\vec{c} = \frac{\sqrt{3}}{2}\vec{b} + \frac{\sqrt{3}}{2}\vec{c}$$

Hence $\vec{a}\,.\,\vec{c} = \frac{\sqrt{3}}{2}$ and $\vec{a}\,.\vec{b} = -\frac{\sqrt{3}}{2}$

$$\vec{a}\,.\,\vec{b} = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \ \cos\theta = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \ \theta = \frac{5\pi}{6}$$

11. (3) $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}, \ \vec{b} = \hat{i} + \hat{j}$

$$\Rightarrow \ \vec{a}\times\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$$

$$\Rightarrow \ |\vec{a}\times\vec{b}| = 3$$

Now, $|(\vec{a}\times\vec{b})\times\vec{c}| = 3$

$$\Rightarrow \ 3\cdot|\vec{c}|\cdot\sin 30^\circ = 3$$

$$\Rightarrow \ |\vec{c}| = 2$$

Also, $|\vec{c} - \vec{a}| = 3$

$$\Rightarrow \ c^2 + a^2 - 2\vec{a}\cdot\vec{c} = 9$$

$\Rightarrow \quad 4 + 9 - 2\vec{a}\cdot\vec{c} = 9$

$\Rightarrow \quad \vec{a}\cdot\vec{c} = 2$

12. (2) Let $\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$

Vectors $\vec{u}$, $\vec{a}$ and $\vec{b}$ are coplanar.

$$\therefore \quad \begin{vmatrix} x & y & z \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 2x - y + z = 0 \qquad$ (i)

Also, it is given that $\vec{u} \perp \vec{a}$.

$\therefore \quad \vec{u}\cdot\vec{a} = 0$

$\Rightarrow \quad 2x + 3y - z = 0 \qquad$ (ii)

Also, $\vec{u}\ \vec{b} = 24$.

$\therefore \quad y + z = 24 \qquad$ (iii)

Solving (i), (ii) and (iii), we get

$x = -4, y = 8$ and $z = 16$

$|\vec{u}|^2 = x^2 + y^2 + z^2$

$= 16 + 64 + 256 = 336$

JEE Advanced

Single Correct Answer Type

1. (3) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 1$ is possible only when $|\vec{a}\times\vec{b}| = |\vec{c}\times\vec{d}| = 1$ and $(\vec{a}\times\vec{b}) \parallel (\vec{c}\times\vec{d})$.

Since $\vec{a}\cdot\vec{c} = \frac{1}{2}$ and if $\vec{b} \parallel \vec{d}$, then $|\vec{c}\times\vec{d}| \neq 1$

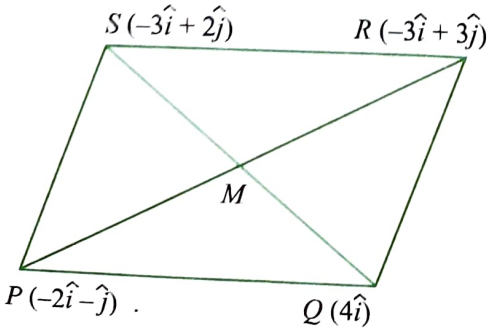
2. (2) Angle between vectors $\vec{AB}$ and $\vec{AD}$ is given by

$$\cos\theta = \frac{\vec{AB}\cdot\vec{AD}}{|\vec{AB}|\cdot|\vec{AD}|} = \frac{-2 + 20 + 22}{\sqrt{4 + 100 + 121}\sqrt{1 + 4 + 4}}$$

$$= \frac{8}{9}$$

$\Rightarrow \quad \cos\alpha = \cos(90° - \theta) = \sin\theta = \frac{\sqrt{17}}{9}$

3. (1)



Evaluating midpoint of PR and QS which gives $M \equiv \left[\frac{\hat{i}}{2} + \hat{j}\right]$, same for both.

$\overrightarrow{PQ} = \overrightarrow{SR} = 6\hat{i} + \hat{j}$

$\overrightarrow{PS} = \overrightarrow{QR} = -\hat{i} + 3\hat{j}$

$\Rightarrow \quad \overrightarrow{PQ}\cdot\overrightarrow{PS} \neq 0$

$\overrightarrow{PQ} \parallel \overrightarrow{SR}$, $\overrightarrow{PS} \parallel \overrightarrow{QR}$ and $|\overrightarrow{PQ}| = |\overrightarrow{SR}|, |\overrightarrow{PS}| = |\overrightarrow{QR}|$

Hence, $PQRS$ is a parallelogram but not rhombus or rectangle.

4. (3) $\vec{v} = \lambda\vec{a} + \mu\vec{b}$

$= \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

Projection of $\vec{v}$ on $\vec{c}$

$$\frac{\vec{v}\cdot\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}}$$

or $\quad \frac{[(\lambda+\mu)\hat{i} + (\lambda-\mu)\hat{j} + (\lambda+\mu)\hat{k}]\cdot(\hat{i} - \hat{j} - \hat{k})}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

or $\quad \lambda + \mu - \lambda + \mu - \lambda - \mu = 1$

or $\quad \mu - \lambda = 1$

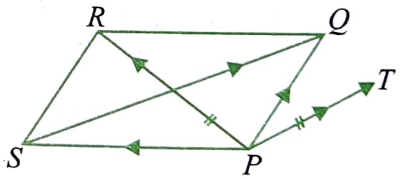
or $\quad \lambda = \mu - 1$

$\Rightarrow \quad \vec{v} = (\mu - 1)(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

$= (2\mu - 1)\hat{i} - \hat{j} + (2\mu - 1)\hat{k}$

At $\mu = 2$, $\vec{v} = 3\hat{i} - \hat{j} + 3\hat{k}$

5. (3)



Area of base ($PQRS$)

$$= \frac{1}{2}|\overrightarrow{PR}\times\overrightarrow{SQ}| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & -4 \end{vmatrix}$$

$$= \frac{1}{2}|-10\hat{i} + 10\hat{j} - 10\hat{k}|$$

$$= 5|\hat{i} - \hat{j} + \hat{k}| = 5\sqrt{3}$$

Height = Projection of PT on $\hat{i} - \hat{j} + \hat{k}$

$$= \left|\frac{1 - 2 + 3}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$$

$\therefore \quad$ Volume $= (5\sqrt{3})\left(\frac{2}{\sqrt{3}}\right) = 10$ cu. unit

6. (4) $\overrightarrow{OP}\cdot\overrightarrow{OQ}+\overrightarrow{OR}\cdot\overrightarrow{OS}=\overrightarrow{OR}\cdot\overrightarrow{OP}+\overrightarrow{OQ}\cdot\overrightarrow{OS}$

$\Rightarrow \quad \overrightarrow{OP}\cdot(\overrightarrow{OQ}-\overrightarrow{OR})=\overrightarrow{OS}\cdot(\overrightarrow{OQ}-\overrightarrow{OR})$

$\Rightarrow \quad (\overrightarrow{OP}-\overrightarrow{OS}).(\overrightarrow{RQ})=0 \Rightarrow \overrightarrow{SP}\cdot\overrightarrow{RQ}=0$

$\Rightarrow \quad \overrightarrow{SP}\perp\overrightarrow{RQ}$

Similarly, $\overrightarrow{SR}\perp\overrightarrow{OP}$ and $\overrightarrow{SQ}\perp\overrightarrow{PR}$.

Hence, S is orthocenter.

Multiple Correct Answers Type

1. (1), (4)

Any vector in the plane of $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ is

$$\vec{r}=\lambda(\hat{i}+\hat{j}+2\hat{k})+\mu(\hat{i}+2\hat{j}+\hat{k})$$

$$=(\lambda+\mu)\hat{i}+(\lambda+2\mu)\hat{j}+(2\lambda+\mu)\hat{k}$$

Also $\vec{r}$ is perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow \quad \vec{r}\cdot\vec{c}=0$

$\Rightarrow \quad \lambda+\mu=0$

Possible vectors are $\hat{j}-\hat{k}$ or $-\hat{j}+\hat{k}$

2. (1), (2), (3)

According to the question

$$\vec{x}\cdot\vec{z}=\vec{x}\cdot\vec{y}=\vec{y}\cdot\vec{z}=\sqrt{2}\cdot\sqrt{2}\cdot\cos\frac{\pi}{3}=1$$

Given $\vec{a}$ is perpendicular to $\vec{x}$ and $\vec{y}\times\vec{z}$

$\therefore \quad \vec{a}=\lambda_1(\vec{x}\times(\vec{y}\times\vec{z}))$

$\Rightarrow \quad \vec{a}=\lambda_1((\vec{x}\cdot\vec{z})\vec{y}-(\vec{x}\cdot\vec{y})\vec{z})$

$\Rightarrow \quad \vec{a}=\lambda_1(\vec{y}-\vec{z}) \qquad (1)$

Now $\vec{a}\cdot\vec{y}=\lambda_1(\vec{y}\cdot\vec{y}-\vec{y}\cdot\vec{z})=\lambda_1(2-1)$

$\Rightarrow \quad \lambda_1=\vec{a}\cdot\vec{y} \qquad (2)$

From (1) and (2), $\vec{a}=(\vec{a}\cdot\vec{y})(\vec{y}-\vec{z})$

Similarly, $\vec{b}=(\vec{b}\cdot\vec{z})(\vec{z}-\vec{x})$

Now, $\vec{a}\cdot\vec{b}=(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})[(\vec{y}-\vec{z})\cdot(\vec{z}-\vec{x})]$

$$=(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})[1-1-2+1]$$

$$=-(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})$$

3. (1), (3), (4)

$\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \quad \vec{b}+\vec{c}=-\vec{a}$

$\Rightarrow \quad |\vec{b}|^2+|\vec{c}|^2+2\vec{b}\cdot\vec{c}=|\vec{a}|^2$

$\Rightarrow \quad 48+|\vec{c}|^2+48=144$

$\Rightarrow \quad |\vec{c}|^2=48$

$\Rightarrow \quad |\vec{c}|=4\sqrt{3}$

$\therefore \quad \frac{|\vec{c}|^2}{2}-|\vec{a}|=24-12=12$

$$\frac{|\vec{c}|^2}{2}+|\vec{a}|=36$$

Further,

$\vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{c}|^2$

$\Rightarrow \quad 144+48+2\vec{a}\cdot\vec{b}=48$

$\Rightarrow \quad \vec{a}\cdot\vec{b}=-72$

$\because \quad \vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \quad \vec{a}\times\vec{b}+\vec{a}\times\vec{c}=0$

$\therefore \quad |\vec{a}\times\vec{b}+\vec{c}\times\vec{a}|$

$$=2|\vec{a}\times\vec{b}|$$

$$=2\sqrt{a^2b^2-(\vec{a}\cdot\vec{b})^2}$$

$$=2\sqrt{(144)(48)-(72)^2}=48\sqrt{3}$$

4. (2), (3)

$|\hat{u}\times\vec{v}|=1$

$\Rightarrow \quad |\vec{v}|\sin\theta=1$, where θ is angle between $\hat{u}$ and $\vec{v}$

Also $\hat{w}.(\hat{u}\times\vec{v})=1$

$\Rightarrow \quad |\hat{w}||\hat{u}||\vec{v}|\sin\theta\cos\alpha=1$, where α is angle between $\hat{w}$ and $(\hat{u}\times\vec{v})$

$\Rightarrow \quad 1\cdot1(1)\cos\alpha=1$

$\Rightarrow \quad \alpha=0$

$\Rightarrow \quad \hat{u}\times\vec{v}=\lambda\hat{w}$ where $\lambda>0$

$$\Rightarrow \quad \begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}=\frac{\lambda}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$$

$$\Rightarrow \quad (u_2v_3 - u_3v_2)\,\hat{i} + (u_3v_1 - u_1v_3)\,\hat{j} + (u_1v_2 - u_2v_1)\,\hat{k} = \frac{\lambda}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$$

$\vec{v}$ is a vector such that $(\hat{u} \times \vec{v})$ is parallel to $\hat{w}$

$$u_3 = 0 \Rightarrow u_2v_3 = \frac{\lambda}{\sqrt{6}} \text{ and } -u_1v_3 = \frac{\lambda}{\sqrt{6}}$$

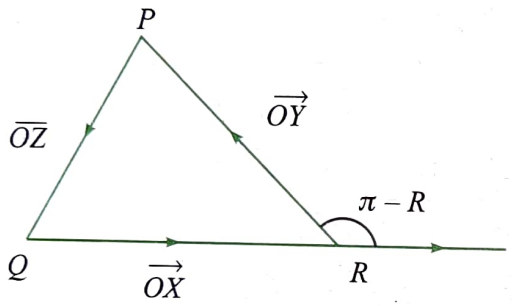
$$\Rightarrow \quad |u_2| = |u_1|$$

$$u_2 = 0 \Rightarrow -u_3v_2 = \frac{\lambda}{\sqrt{6}} \text{ and } u_1v_2 = \frac{2\lambda}{\sqrt{6}}$$

$$\Rightarrow \quad |u_1| = 2|u_3|$$

Linked Comprehension Type

For Problems 1 and 2



1. (1) $\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| = \left|\overrightarrow{OX}\right|\left|\overrightarrow{OY}\right|\sin(\pi - R)$

$= \sin R = \sin(P + Q)$

2. (2) $-(\cos P + \cos Q + \cos R)$

$= \overrightarrow{OX} \cdot \overrightarrow{OY} + \overrightarrow{OY} \cdot \overrightarrow{OZ} + \overrightarrow{OZ} \cdot \overrightarrow{OX}$

$$= \frac{(\overrightarrow{OX} + \overrightarrow{OY} + \overrightarrow{OZ})^2 - (|\overrightarrow{OX}|^2 + |\overrightarrow{OY}|^2 + |\overrightarrow{OZ}|^2)}{2}$$

$$\geq -\frac{3}{2}$$

Matrix Match Type

1. a → r; b → s; c → p; d → q

a. $[\vec{a}\ \vec{b}\ \vec{c}] = 2$

$\Rightarrow [2\vec{a} \times \vec{b} \quad 3\vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = 6[\vec{a}\ \vec{b}\ \vec{c}]^2$

$= 6 \times 4 = 24$

b. $[\vec{a}\ \vec{b}\ \vec{c}] = 5$

$\Rightarrow [3(\vec{a} + \vec{b})\ \vec{b} + \vec{c}\ \ 2(\vec{c} + \vec{a})]$

$= 6[(\vec{a} + \vec{b})\ \vec{b} + \vec{c}\ (\vec{c} + \vec{a})] = 12[\vec{a}\ \vec{b}\ \vec{c}] = 60$

c. Given $\frac{1}{2}|\vec{a} \times \vec{b}| = 20$

Now $\frac{1}{2}|(2\vec{a} + 3\vec{b}) \times (\vec{a} - \vec{b})|$

$= \frac{1}{2}|-2(\vec{a} \times \vec{b}) - 3(\vec{a} \times \vec{b})|$

$= \frac{5}{2} \times 40 = 100$

d. Given $|\vec{a} \times \vec{b}| = 30 \Rightarrow |(\vec{a} + \vec{b}) \times \vec{a}| = |\vec{b} \times \vec{a}| = 30$

2. (1)

b. $(\vec{a_k} \times \vec{a_{k+1}}) = r^2 \sin\frac{2\pi}{n}$

$\vec{a_k} \cdot \vec{a_{k+1}} = r^2 \cos\frac{2\pi}{n}$

Given $\left|\sum_{k=1}^{n-1} \vec{a_k} \times \vec{a_{k+1}}\right| = \left|\sum_{k=1}^{n-1} a_k \cdot a_{k+1}\right|$

$$\Rightarrow \quad r^2(n-1)\sin\frac{2\pi}{n} = r^2(n-1)\cos\frac{2\pi}{n}$$

$$\Rightarrow \quad \tan\frac{2\pi}{n} = 1$$

$$\Rightarrow \quad \frac{2\pi}{n} = k\pi + \frac{\pi}{4},\ k \in Z$$

$$\Rightarrow \quad n = \frac{8}{4k+1}$$

$\Rightarrow \quad n = 8$ (when $k = 0$)

Note: Solutions of the remaining parts are given in their respective chapters.

3. a → p, q

Projection of $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$.

So, $\left|(\alpha\hat{i} + \beta\hat{j}) \cdot \left(\frac{\sqrt{3}\hat{i} + \hat{j}}{2}\right)\right| = \sqrt{3}$

$$\Rightarrow \quad \sqrt{3}\alpha + \beta = \pm 2\sqrt{3}$$

$$\Rightarrow \quad \sqrt{3}\alpha + \left(\frac{\alpha - 2}{\sqrt{3}}\right) = \pm 2\sqrt{3}$$

$$\Rightarrow \quad 3\alpha + \alpha - 2 = \pm 6$$

$$\Rightarrow \quad 4\alpha = 8, -4$$

$$\Rightarrow \quad \alpha = 2, -1$$

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (5) $E = (2\vec{a} + \vec{b}) \cdot [|\vec{a}|^2\ \vec{b} - (\vec{a} \cdot \vec{b})\vec{a} - 2(\vec{a} \cdot \vec{b})\ \vec{b} + 2|\vec{b}|^2\ \vec{a}]$

$$\vec{a} \cdot \vec{b} = \frac{2-2}{\sqrt{70}} = 0$$

$|\vec{a}| = 1$

$|b| = 1$

$E = (2\vec{a} + \vec{b}) \cdot [2|\vec{b}|^2\ \vec{a} + |\vec{a}|^2\ \vec{b}]$

$$= 4|\vec{a}|^2|\vec{b}|^2 + |\vec{a}|^2(\vec{a}\cdot\vec{b}) + 2|\vec{b}|^2(\vec{b}\cdot\vec{a}) + |\vec{a}|^2|\vec{b}|^2$$

$$= 5|\vec{a}|^2|\vec{b}|^2 = 5$$

2. (9) $\vec{r}\times\vec{b} = \vec{c}\times\vec{b}$

Taking cross product with $\vec{a}$

$$\vec{a}\times(\vec{r}\times\vec{b}) = \vec{a}\times(\vec{c}\times\vec{b})$$

or $(\vec{a}\cdot\vec{b})\vec{r} - (\vec{a}\cdot\vec{r})\vec{b} = \vec{a}\times(\vec{c}\times\vec{b})$

or $\vec{r} = -3\hat{i} + 6\hat{j} + 3\hat{k}$

$\Rightarrow$ $\vec{r}\cdot\vec{b} = 3 + 6 = 9$

3. (3) As $|\vec{a}-\vec{b}|^2 + |\vec{b}-\vec{c}|^2 + |\vec{c}-\vec{a}|^2$

$$= 3(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2) - |\vec{a}+\vec{b}+\vec{c}|^2$$

$\Rightarrow$ $3\times 3 - |\vec{a}+\vec{b}+\vec{c}| = 9$

or $|\vec{a}+\vec{b}+\vec{c}| = 0$

or $\vec{a}+\vec{b}+\vec{c} = 0$

or $\vec{b}+\vec{c} = -\vec{a}$

$\Rightarrow$ $|2\vec{a} + 5(\vec{b}+\vec{c})| = |-3\vec{a}| = 3|\vec{a}| = 3.$

4. (4) $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

Also, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 1/2$

$$\vec{a}\times\vec{b} + \vec{b}\times\vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$$

$\Rightarrow$ $\vec{a}\cdot(\vec{b}\times\vec{c}) = p + q(\vec{a}\cdot\vec{b}) + r(\vec{a}\cdot\vec{c})$

$\therefore$ $p + \frac{q}{2} + \frac{r}{2} = [\vec{a}\,\vec{b}\,\vec{c}]$ (1)

Similarly, taking dot product with vector $\vec{b}$, we get

$$\frac{p}{2} + q + \frac{r}{2} = 0 \quad (2)$$

And, taking dot product with vector $\vec{c}$, we get

$$\frac{p}{2} + \frac{q}{2} + r = [\vec{a}\,\vec{b}\,\vec{c}] \quad (3)$$

Solving, (1), (2) and (3), we get

$$p = r = -q$$

$\Rightarrow$ $\frac{p^2 + 2q^2 + r^2}{q^2} = 4$

5. (3) $\vec{c} = x\vec{a} + y\vec{b} + \vec{a}\times\vec{b}$ (i)

and $\vec{a}\ \vec{b} = 0,\ |\vec{c}| = 2,\ |\vec{a}| = |\vec{b}| = 1$

Let vector $\vec{c}$ make angle α with vectors $\vec{a}$ and $\vec{b}$.

$\therefore$ $\vec{c}\ \vec{a} = \vec{c}\ \vec{b} = 2\cos\alpha$

Taking dot product of equation (i) with $\vec{a}$ and $\vec{b}$, we have

$\Rightarrow$ $x = y = 2\cos\alpha$

Now, $|\vec{c}|^2 = x^2 + y^2 + |\vec{a}\times\vec{b}|^2 = 2(4\cos^2\alpha) + 1$

$\Rightarrow$ $4 = 8\cos^2\alpha + 1$

$\Rightarrow$ $8\cos^2\alpha = 3$

Chapter 4

Concept Application Exercises

Exercise 4.1

1. Suppose P divides QR in the ratio $\lambda : 1$.
Then the coordinates of P are
$$\frac{5\lambda+2}{\lambda+1}, \frac{\lambda+2}{\lambda+1} \text{ and } \frac{-2\lambda+1}{\lambda+1}.$$
It is given that the x-coordinate of P is 4. Therefore,
$$\frac{5\lambda+2}{\lambda+1}=4 \text{ or } \lambda=2$$
So z-coordinate of P is $\frac{-2\lambda+1}{\lambda+1}=\frac{-4+1}{2+1}=-1$

2. The direction ratios of the $\vec{r}$ are 2, –3 and 6.
Therefore, its direction cosines are $l=\frac{2}{7}$, $m=\frac{-3}{7}$ and $n=\frac{6}{7}$. Therefore,
$$\vec{r}=|\vec{r}|(l\hat{i}+m\hat{j}+n\hat{k})=21\left(\frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right)$$
$$=6\hat{i}-9\hat{j}+18\hat{k}$$

3. Since $\overrightarrow{OP}$ has projections $\frac{13}{5}, \frac{19}{5}$ and $\frac{26}{5}$ on the coordinate axes, $\overrightarrow{OP}=\frac{13}{5}\hat{i}+\frac{19}{5}\hat{j}+\frac{26}{5}\hat{k}$. Suppose P divides the join of $Q(2, 2, 4)$ and $R(3, 5, 6)$ in the ratio $\lambda : 1$. Then the position vector of P is
$$\left(\frac{3\lambda+2}{\lambda+1}\right)\hat{i}+\left(\frac{5\lambda+2}{\lambda+1}\right)\hat{j}+\left(\frac{6\lambda+4}{\lambda+1}\right)\hat{k}$$
$$\therefore \quad \frac{13}{5}\hat{i}+\frac{19}{5}\hat{j}+\frac{26}{5}\hat{k}$$
$$=\left(\frac{3\lambda+2}{\lambda+1}\right)\hat{i}+\left(\frac{5\lambda+2}{\lambda+1}\right)\hat{j}+\left(\frac{6\lambda+4}{\lambda+1}\right)\hat{k}$$
Thus, we have
$$\frac{3\lambda+2}{\lambda+1}=\frac{13}{5}, \frac{5\lambda+2}{\lambda+1}=\frac{19}{5} \text{ and } \frac{6\lambda+4}{\lambda+1}=\frac{26}{5}$$
$\Rightarrow \quad 2\lambda=3$ or $\lambda=3/2$
Hence, P divides QR in the ratio 3 : 2.

4. The direction cosines of $\overrightarrow{OP}$ are $-\frac{1}{3}, \frac{2}{3}$ and $-\frac{2}{3}$.
Hence, $\overrightarrow{OP}=|\overrightarrow{OP}|(l\hat{i}+m\hat{j}+n\hat{k})$
$$=3\left(-\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}-\frac{2}{3}\hat{k}\right)$$
$$=-1\hat{i}+2\hat{j}-2\hat{k}$$
So, the coordinates of P are –1, 2 and –2.

5. According to the question, $\frac{a+2}{6}=\frac{b-1}{2}=\frac{c+8}{3}=\lambda$
or $\quad a=6\lambda-2, b=2\lambda+1, c=3\lambda-8$

6. Since $\frac{a}{(1/bc)}=\frac{b}{(1/ca)}=\frac{c}{(1/ab)}$, lines are parallel.

7. Eliminating n, we have $(2l+m)(l-m)=0$.
When $2l+m=0$, then $\frac{l}{1}=\frac{m}{-2}=\frac{n}{-2}$.
When $l-m=0$, then $\frac{l}{1}=\frac{m}{1}=\frac{n}{-2}$. Therefore,
Direction ratios are 1, –2, 1 and 1, 1, and –2
$$\cos\theta=\frac{\sum l_1l_2}{\sqrt{\left(\sum l_1^2\right)\left(\sum l_2^2\right)}}=-\frac{1}{2}$$
or $\quad \theta=120^\circ=2\pi/3$

Exercise 4.2

1. Line is passing through the point (1, 2, 3) and parallel to the line $\vec{r}=\hat{i}-\hat{j}+2\hat{k}+\lambda(\hat{i}-2\hat{j}+3\hat{k})$ or parallel to the vector $\hat{i}-2\hat{j}+3\hat{k}$. Hence, equation of line is
$$\frac{x-1}{1}=\frac{y-2}{-2}=\frac{z-3}{3}$$
It meets xy-plane, where $z=0$
Then from the equation of line, we have
$$\frac{x-1}{1}=\frac{y-2}{-2}=\frac{0-3}{3}$$
$\Rightarrow \quad x=0, y=4.$
Hence, line meets xy-plane at (0, 4, 0).

2. Since line is passing through the points $A(1, 2, 3)$ and $B(-1, 0, 4)$, it is along the vector $\vec{AB}=-2\hat{i}-2\hat{j}+\hat{k}$.
Hence, equation of line is
$$\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(-2\hat{i}-2\hat{j}+\hat{k})$$
or $\quad \vec{r}=-\hat{i}+4\hat{k}+\lambda(-2\hat{i}-2\hat{j}+\hat{k})$

$$\Rightarrow \quad \frac{x-1}{-2}=\frac{y-2}{-2}=\frac{z-3}{1}$$

or $$\frac{x+1}{-2}=\frac{y-0}{-2}=\frac{z-4}{1}$$

3. The given line is $-6x-2=3y+1=2z-2$

or $$\frac{x+(1/3)}{-1/6}=\frac{y+(1/3)}{1/3}=\frac{z-1}{1/2}$$

The direction ratios are $-\frac{1}{6}, \frac{1}{3}$ and $\frac{1}{2}$ or -1, 2 and 3.

The required equation is $\frac{x-2}{-1}=\frac{y+1}{2}=\frac{z+1}{3}$

4. The line through point $(-1, 2, 3)$ is perpendicular to the lines $\frac{x}{2}=\frac{y-1}{-3}=\frac{2+2}{-2}$ and $\frac{x+3}{-1}=\frac{y+3}{2}=\frac{z-1}{3}$.

Therefore, the line is along the vector $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -2 \\ -1 & 2 & 3 \end{vmatrix}$

or $-5\hat{i}-4\hat{j}+\hat{k}$.

Hence, equation of the line is $\frac{x+1}{5}=\frac{y-2}{4}=\frac{z-3}{-1}$.

5. Intersection point of the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-4}{5}=\frac{y-1}{2}=z$$

is $(-1, -1, -1)$ (on solving). Therefore, the equation of the line passing through the points $(-1, -1, -1)$ and $(2, 1, -2)$ is $\frac{x+1}{3}=\frac{y+1}{2}=\frac{z+1}{-1}$.

6. The line is along the vector $3\hat{i}+\hat{j}$ which is perpendicular to the z-axis as $(3\hat{i}+\hat{j})\cdot\hat{k}=0$.

7. The lines are $\frac{x}{3}=\frac{y}{2}=\frac{z}{-6}$ and $\frac{x}{2}=\frac{y}{-12}=\frac{z}{-3}$.

Since $a_1a_2+b_1b_2+c_1c_2=6-24+18=0$, we have

$\theta=90°$

8. The lines are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$.

Hence, $-3(3k)+2k(1)+2(-5)=0$ or $k=-\frac{10}{7}$.

9. Eliminating t from the given equations, we get equation of the path.

$$\frac{x}{2}=\frac{y}{4}=\frac{z}{4}$$

or $$\frac{x}{1}=\frac{y}{-2}=\frac{z}{2}$$

Thus, the path of the rocket represents a straight line passing through the origin.

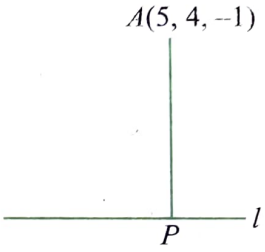
For $t=10$ s, we have

$x=20, y=-40$ and $z=40$

and $|\vec{r}|=|\overrightarrow{OM}|=\sqrt{x^2+y^2+z^2}$

$=\sqrt{400+1600+1600}=60$ km

10. Let P be the foot of the perpendicular from the point $A(5, 4, -1)$ to the line l whose equation is $\vec{r}=\hat{i}+\lambda(2\hat{i}+9\hat{j}+5\hat{k})$.



The coordinates of any point on the line are given by

$x=1+2\lambda$,

$y=9\lambda$ and $z=5\lambda$.

The coordinates of P are given by $1+2\lambda$, 9λ and 5λ for some value of λ.

The direction ratios of AP are $1+2\lambda-5$, $9\lambda-4$ and $5\lambda-(-1)$ or $2\lambda-4$, $9\lambda-4$ and $5\lambda+1$.

Also, the direction ratios of l are 2, 9 and 5.

Since $AP \perp l$, $a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow \quad 2(2\lambda-4)+9(9\lambda-4)+5(5\lambda+1)=0$

or $\quad 4\lambda-8+81\lambda-36+25\lambda+5=0$

or $\quad 110\lambda-39=0$ or $\lambda=39/110$

Now, $AP^2=(1+2\lambda-5)^2+(9\lambda-4)^2+(5\lambda-(-1))^2$

$=(2\lambda-4)^2+(9\lambda-4)^2+(5\lambda+1)^2$

$=4\lambda^2-16\lambda+16+81\lambda^2-72\lambda+16+25\lambda^2+10\lambda+1=110\lambda^2-78\lambda+33$

$$=110\left(\frac{39}{110}\right)^2-78\left(\frac{39}{110}\right)+33$$

$$=\frac{39^2-78\times39+33\times110}{110}=\frac{2109}{110}$$

or $$AP=\sqrt{\frac{2109}{110}}$$

11. Let the image of point $A(1, 2, 3)$ in the line l whose equation is

$$\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}=k \text{ (say)} \qquad \text{(i)}$$

be A'. Then AA' is perpendicular to l and the point of intersection of AA' and l is the midpoint of AA'. Note that M is the foot of perpendicular from A to l.

The coordinates of any point on the given line are of the form $(3k+6, 2k+7, -2k+7)$. Therefore, the coordinates of M are $3k+6$, $2k+7$ and $-2k+7$

for some value of k. The direction ratios of AM are $3k + 6 - 1$, $2k + 7 - 2$ and $-2k + 7 - 3$ or $3k + 5$, $2k + 5$, $-2k + 4$.

Also, the direction ratios of l are 3, 2 and -2.

Since $AM \perp l$, $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

$\Rightarrow \quad 3(3k + 5) + 2(2k + 5) - 2(-2k + 4) = 0$

or $\quad 17k + 17 = 0 \quad$ or $\quad k = -1$

Thus, the coordinates of M are 3, 5 and 9.

Suppose coordinates of A' are x, y and z,

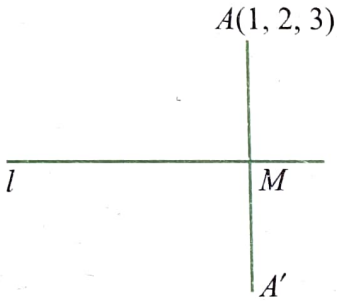
The coordinates of the midpoint of AA' are $\frac{x+1}{2}, \frac{y+2}{2}$ and $\frac{z+3}{2}$.

But the midpoint of AA' is (5, 3, 9). Therefore,

$$\frac{x+1}{2} = 3, \frac{y+2}{2} = 5 \text{ and } \frac{z+3}{2} = 9$$

$\Rightarrow \quad x = 5, y = 8, z = 15$

Thus, the image of A in l is (5, 8, 15).



12. The lines are $\vec{r} = (1-\lambda)\hat{i} + (\lambda - 2)\hat{j} + (3 - 2\lambda)\hat{k}$

and $\quad \vec{r} = (\mu + 1)\hat{i} + (2\mu - 1)\hat{j} - (2\mu + 1)\hat{k}$

or $\quad \vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} - 2\hat{j} - 2\hat{k})$

and $\quad \vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} + 2\hat{j} - 2\hat{k})$.

Line (i) passes through the point $(x_1, y_1, z_1) \equiv (1, -2, 3)$ and is parallel to the vector

$$a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \equiv -\hat{i} - 2\hat{j} - 2\hat{k}.$$

Line (ii) passes through the point $(x_2, y_2, z_2) \equiv (1, -1, -1)$ and is parallel to the vector

$$a_2\hat{i} + b_2\hat{j} + c_2\hat{k} \equiv \hat{i} + 2\hat{j} - 2\hat{k}.$$

Hence, the shortest distance between the lines using the formula $\dfrac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$ is

$$\frac{\begin{vmatrix} 1-1 & -1+2 & -1-3 \\ -1 & -2 & -2 \\ 1 & 2 & -2 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -2 \\ 1 & 2 & -2 \end{vmatrix}} = \frac{4}{\sqrt{80}} = \frac{1}{\sqrt{5}}$$

13. We have line of intersection of planes $x + y + 2z - 3 = 0$, $2x + 3y + 4z - 4 = 0$.

Vector along line of intersection is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{vmatrix} = -2\hat{i} + \hat{k}$$

For any point on this line let $z = 0$, so we get $x = 5$ and $y = -2$

Therefore, equation of line in symmetric form is

$$\frac{x-5}{-2} = \frac{y+2}{0} = \frac{z}{1}$$

Equation of z-axis is $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$

Hence, the shortest distance between these lines is

$$\frac{\begin{vmatrix} 5-0 & -2-0 & 0-0 \\ -2 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}} = 2$$

14. $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4} = \lambda$

$\Rightarrow \quad x = 2\lambda + 1, y = 3\lambda - 1$ and $z = 4\lambda + 1$.

$$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = \mu$$

$\Rightarrow \quad x = 3 + \mu, y = k + 2\mu$ and $z = \mu$.

Since the above lines intersect, we have

$2\lambda + 1 = 3 + \mu$ (i)

$3\lambda - 1 = 2\mu + k$ (ii)

$\mu = 4\lambda + 1$ (iii)

Solving (i) and (iii) and putting the value of λ and μ in (ii), $k = 9/2$.

15. Let equation of the line ℓ_1 be $\vec{r} = \vec{a} + \lambda\vec{b}$ and equation of the line ℓ_2 be $\vec{r} = \vec{c} + \mu\vec{d}$, where $\vec{a} - \vec{c}$, $\vec{b}$ and $\vec{d}$ are non-coplanar.

Let the position vectors of points P and Q be $\vec{a}+\lambda_1\vec{b}$ and $\vec{a}+\lambda_2\vec{b}$, respectively.

Let the position vectors of points R and S be $\vec{c}+\mu_1\vec{d}$ and $\vec{c}+\mu_2\vec{d}$, respectively.

Then the lines PR and QS are parallel if and only if

$$\vec{c}-\vec{a}+\mu_1\vec{d}-\lambda_1\vec{b}=k\ (\vec{c}-\vec{a}+\mu_2\vec{d}-\lambda_2\vec{b})$$

i.e., $(1-k)\ (\vec{c}-\vec{a})\ +(\mu_1-k\mu_2)\ \vec{d}$

$$-(\lambda_1-k\lambda_2)\ \vec{b}\ =\ \vec{o}$$

$\therefore\quad 1-k=0,\ \mu_1-k\mu_2=0,\ \lambda_1-k\lambda_2=0$

i.e., $\mu_1=\mu_2$ and $\lambda_1=\lambda_2$ which is not possible

Therefore, PR can not be parallel to QS.

Exercise 4.3

1. The angle between a line and a plane is complement of the angle between the line and the normal of the plane, i.e., 3, 2, 4 and normal 2, 1, –3. Therefore,

$$\cos\theta=\frac{6+2-12}{\sqrt{29}\cdot\sqrt{14}}=-\frac{4}{\sqrt{406}}$$

$$\theta=\cos^{-1}\ (-4/\sqrt{406})$$

$$\phi=90^\circ-\theta$$

$$=90^\circ-\cos^{-1}\ (-4/\sqrt{406})$$

$$=\sin^{-1}\ (-4/\sqrt{406})$$

2. The line is along the vector $\vec{a}=-3\hat{i}+2\hat{j}+\hat{k}$ and plane is normal to the vector $\vec{b}=\hat{i}+\hat{j}+\hat{k}$.

Since $\vec{a}\cdot\vec{b}=0$, the line is parallel to the plane.

Hence, the distance between the line and the plane is the distance of point $(-1, 3, 2)$ from the plane,

$$\frac{|-1+3+2+3|}{\sqrt{1+1+1}}=\frac{7}{\sqrt{3}}$$

3. Any point on the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=\lambda$ is $(3\lambda+2, 4\lambda-1, 12\lambda+2)$.

This lies on $x-y+z=5$.

If $3\lambda+2-4\lambda+1+12\lambda+2=5$ or $\lambda=0$, then the point is $(2, -1, 2)$.

Its distance from $(-1, -5, -10)$ is

$$\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2}$$

$$=\sqrt{9+16+144}=13$$

4. Given planes are

$$x+2y+3z=5 \qquad \text{(i)}$$

$$3x+3y+z=0 \qquad \text{(ii)}$$

Normal to planes (i) and (ii) are $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}+3\hat{j}+\hat{k}$, respectively.

Required plane is perpendicular to planes (i) and (ii), i.e., to perpendicular to the line of intersection of planes.

Vector along the line of intersection of planes (i) and (ii) is

$$\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2&3\\3&3&1\end{vmatrix}=-7\hat{i}+8\hat{j}-3\hat{k}$$

Therefore, normal to given plane is $-7\hat{i}+8\hat{j}-3\hat{k}$.

Plane is passing through the point $(-1, 3, 2)$

Then equation of required plane is

$$-7(x+1)+8(y-3)-3(z-2)=0$$

or $\quad 7x-3y+8z+25=0$

Also the plane is passing through the point $(1, 2, 0)$; hence, the equation of the plane is

$$0\ (x-1)+3\ (y-2)+3\ (z-0)=0$$

or $\quad y+z-2=0.$

5. The equation of any plane through the point $(1, 0, -1)$ is

$$A\ (x-1)+B\ (y-0)+C\ (z+1)=0 \qquad \text{(i)}$$

Since it passes through the point $(3, 2, 2)$, we get

$$2A+2B+3C=0 \qquad \text{(ii)}$$

Since plane (i) is parallel to the line

$$\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$$

we have $1A+(-2)\ B+3C=0 \qquad \text{(iii)}$

From (i) and (iii),

$$A:B:C=4:-1:-2$$

Substituting these values in (i), we get

$$4\ (x-1)-1\ (y-0)-2\ (z+1)=0$$

or $\quad 4x-y-2z-6=0$

6. The required plane is

$$\begin{vmatrix}x-5&y-7&z+3\\4&4&-5\\7&1&3\end{vmatrix}=0$$

or $\quad 17(x-5)-(12+35)(y-7)+(4-28)(z+3)=0$

or $\quad 17x-47y-24z+172=0$

7. Let the equation of a plane containing the line be $l\ (x-1)+m\ (y+2)+nz=0$. Then

$$2l-3m+5n=0 \text{ and } l-m+n=0$$

$$\therefore\quad \frac{l}{2}=\frac{m}{3}=\frac{n}{1}$$

Hence, the plane is $2\ (x-1)+3\ (y+2)+z=0$

or $\quad 2x+3y+z+4=0$

8. Any plane passing through the origin is $a(x-0)+b(y-0)+c(z-0)=0$

This is perpendicular to the given line. Therefore, the normal to the plane is parallel to the given line. Thus,

$$\frac{a}{2}=\frac{b}{-1}=\frac{c}{2}$$

Therefore, the required plane is

$$2(x-0)-1(y-0)+2(z-0)=0$$

or $\quad 2x-y+2z=0$

9. Any plane through $\frac{x-1}{5}=\frac{y+2}{6}=\frac{z-3}{4}$ is

$$A(x-1)+B(y+2)+C(z-3)=0 \qquad \text{(i)}$$

where $5A+6B+4C=0$ (ii)

Also, the plane passes through (4, 3, 7). Therefore,

$$3A+5B+4C=0 \qquad \text{(iii)}$$

By (ii) and (iii), $\frac{A}{4}=\frac{B}{-8}=\frac{C}{7}$

Therefore, the plane is

$$4(x-1)-8(y+2)+7(z-3)=0$$

or $\quad 4x-8y+7z=41.$

10. The given line is $\vec{r}=(\vec{i}+2\vec{j}-\vec{k})+\lambda(\vec{i}-\vec{j}+\vec{k})$

Here, $\vec{b}=\vec{i}-\vec{j}+\vec{k}$ (type $\vec{r}=\vec{a}+\lambda\vec{b}$)

The given plane is $\vec{r}\cdot(2\vec{i}-\vec{j}+\vec{k})=4$

(type $\vec{r}\cdot\vec{n}=p$)

Here $\vec{n}=2\vec{i}-\vec{j}+\vec{k}$

Now $\cos\theta=\dfrac{\vec{n}\cdot\vec{b}}{|\vec{n}||\vec{b}|}$

(If θ is the angle between the line and the normal to the plane)

$$=\frac{(2\vec{i}-\vec{j}+\vec{k})\cdot(\vec{i}-\vec{j}+\vec{k})}{\sqrt{4+1+1}\sqrt{1+1+1}}$$

$$=\frac{2+1+1}{\sqrt{6}\sqrt{3}}=\frac{4}{\sqrt{2}\cdot 3}=\frac{2\sqrt{2}}{3}$$

$$\therefore \quad \theta=\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

11. The plane passes through the point $A(1, 2, 3)$ and is at the maximum distance from point $B(-1, 0, 2)$; then the plane is perpendicular to line AB. Therefore, the direction ratios of the normal to the plane are 2, 2 and 1.

Hence, the equation of the plane is

$$2(x-1)+2(y-2)+1(z-3)=0$$

or $\quad 2x+2y+z=9$

12. Any point on the line $\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z-2}{3}=t$ is $(t+1, -2t-1, 3t+2)$, which lies on the given plane if $t+1+2t+1+6t+4-3=0$ or $t=-1/3$.

Hence, the point of intersection of the line and the plane is $P(2/3, -1/3, 1)$.

Also, if the foot of the perpendicular from $A(1, -1, 2)$ on the plane is $Q(x, y, z)$, then

$$\frac{x-1}{1}=\frac{y+1}{-1}=\frac{z-2}{2}$$

$$=-\frac{(1+1+4-3)}{1+1+4}=-\frac{1}{2}$$

Therefore, $Q(x, y, z)$ is $Q(1/2, -1/2, 1)$.

Hence, the direction ratios of PQ are $\frac{2}{3}-\frac{1}{2}, -\frac{1}{3}+\frac{1}{2}$ and $1-1$ or $\frac{1}{6}, \frac{1}{6}$ and 0.

If the image of point $A(1, -1, 2)$ in the plane is R, then Q is the midpoint of AR. Therefore, point R is $(0, 0, 0)$.

Hence, the direction ratios of PR or the image of the line in the plane are 2/3, – 1/3 and 1.

13. The equation of the plane parallel to

$$x-2y+2z=5 \text{ is } x-2y+2z+k=0. \qquad \text{(i)}$$

Now, according to the equation,

$$\frac{1-4+6+k}{\sqrt{9}}=\pm 1$$

$$k+3=\pm 3 \quad \Rightarrow \quad k=0 \text{ or } -6$$

Then $x-2y+2z-6=0$ or $x-2y+2z=6$

14. Plane which is equally inclined to the given planes is parallel to the angle bisector of the given planes.

Now the angle bisector of the given planes is

$$\frac{x-2y+2z-3}{3}=\pm\frac{8x-4y+z-7}{9}.$$

$$5x+2y-5z+2=0$$

and $\quad 11x-10y+7z-16=0.$

The equation of the required planes are $5x+2y-5z+p=0$ and $11x-10y+7z+q=0$.

Since both are passing through point $(1, 2, 3)$, $p=6$ and $q=12$.

The planes are $5x+2y-5z+6=0$ and $11x-10y+7z+12=0$

15. The image of the plane $x-2y+2z-3=0$ (i)
in the plane $x+y+z-1=0$ (ii)
passes through the line of intersection of the given planes.
Therefore, the equation of such a plane is
$(x-2y+2z-3)+t(x+y+z-1)=0, \quad t\in R$
$(1+t)x+(-2+t)y+(2+t)z-3-t=0$ (iii)
Now plane (ii) makes the same angle with plane (i) and image plane (iii). Thus,

$$\frac{1-2+2}{3\sqrt{3}}=\pm\frac{1+t-2+t+2+t}{\sqrt{3}\sqrt{(t+1)^2+(t-2)^2+(2+t)^2}}$$

$$\frac{1}{3}=\pm\frac{3t+1}{\sqrt{3t^2+2t+9}}$$

$$3t^2+2t+9=9(9t^2+6t+1)$$

$$3t^2+2t+9=81t^2+54t+9$$

$$78t^2+52t=0$$

$$t=0 \text{ or } t=-\frac{2}{3}$$

For $t=0$, we get plane (i); hence for image plane, $t=-\frac{2}{3}$
The equation of the image plane is
$3(x-2y+2z-3)-2(x+y+z-1)=0$
or $x-8y+4z-7=0$

16. Plane passes through the points $(2,3,1)$ and $(4,-5,3)$
Therefore, plane is parallel to vector
$(4-2)\hat{i}+(-5-3)\hat{j}+(3-1)\hat{k}=2\hat{i}-8\hat{j}+2\hat{k}$
Also plane is parallel to the x-axis, so it parallel to vector $\hat{i}$

$$\therefore \text{ Normal to the plane } = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -8 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\hat{j}+8\hat{k}$$

Hence, equation of plane through point (2, 3, 1) and normal to vector $2\hat{j}+8\hat{k}$ is
$0(x-2)+2(y-3)+8(z-1)=0$
or $y+4z-7=0$

17. A line parallel to $\vec{c}$ and passing through $A(\vec{a})$ is $\vec{r}=\vec{a}+\lambda\vec{c}$
A point for some λ satisfies plane
$(\vec{a}+\lambda\vec{c})\cdot\hat{n}=d$
or $\lambda=\dfrac{\vec{d}-\vec{a}\cdot\hat{n}}{\vec{c}\cdot\hat{n}}$

$$\therefore \quad \text{Distance} = \frac{|\vec{d}-\hat{a}\cdot\hat{n}|}{|\vec{c}\cdot\hat{n}|}|\vec{c}|$$

18. Given line is the line of intersection of planes $3x-2y+z+3=0$ and $4x-3y+4z+1=0$.
Now vector along line of intersection is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 4 & -3 & 4 \end{vmatrix} = -5\hat{i}-8\hat{j}-\hat{k}$$

Line is parallel to plane $2x-y+mz-2=0$
So plane is perpendicular to vector $-5\hat{i}+8\hat{j}-\hat{k}$.
Thus,
$(2)(-5)+(-1)(-8)+(m)(-1)=0$
or $m=-2$

19. Given lines are $\dfrac{x-a+d}{\alpha-\delta}=\dfrac{y-a}{\alpha}=\dfrac{z-a-d}{\alpha+\delta}$,
$\dfrac{x-b+c}{\beta-\gamma}=\dfrac{y-b}{\beta}=\dfrac{z-b-c}{\beta+\gamma}$
Now

$$\begin{vmatrix} a-d-b+c & a-b & a+d-b-c \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}$$

$$= \begin{vmatrix} -d+c & a-b & d-c \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix} = 0$$

(Applying $C_1\to C_1-C_2$ and $C_3\to C_3-C_2$)
Hence, the lines are coplanar.
Equation of plane containing these lines is

$$\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2y+z & y-a & z-a-d \\ 0 & \alpha & \alpha+\delta \\ 0 & \beta & \beta+\gamma \end{vmatrix} = 0$$

(Applying $C_1\to C_1+C_3-2C_2$)
or $(x-2y+z)(\alpha\gamma-\beta\delta)=0$
or $x-2y+z=0 \quad [\because (\alpha\gamma\neq\beta\delta)]$

20. The given spheres are
$x^2+y^2+z^2+2x+2y+2z+2=0$ (i)
and $x^2+y^2+z^2+x+y+z-(1/4)=0$ (ii)
The required plane is
$(2x-x)+(2y-y)+(2z-z)+2+\frac{1}{4}=0$
or $4x+4y+4z+9=0$

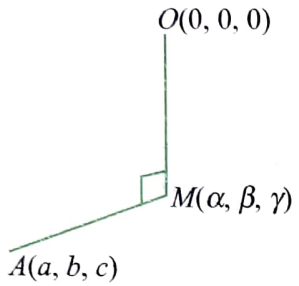
21. The radius of the sphere is 5

The given plane is $x + y - z = 4\sqrt{3}$

The length of the perpendicular from the centre $(0, 0, 0)$ of the sphere on the plane $= \dfrac{4\sqrt{3}}{\sqrt{1+1+1}} = 4$

Hence, radius of the circular section is $\sqrt{25-16} = \sqrt{9} = 3$

22.



Variable plane is passing through the point $A(a, b, c)$ and foot of perpendicular from origin on this variable plane is $M(\alpha, \beta, \gamma)$.

We have to find the locus of point M.

As shown in the figure, vectors $\overline{AM}$ and $\overline{OM}$ are perpendicular. Thus,

$$\overline{AM} \cdot \overline{OM} = 0$$

$$\Rightarrow \quad (\alpha - a)\,\alpha + (\beta - b)\,\beta + (\gamma - c)\,\gamma = 0$$

Therefore, locus of M is

$$x(x-a) + y(y-b) + z(z-c) = 0$$

Exercises

Single Correct Answer Type

1. (2) $x^2 - 5x + 6 = 0$

$\Rightarrow \quad x - 2 = 0, x - 3 = 0$

which represents planes.

2. (3) We have $z = 0$ for the point, where the line intersects the curve. Therefore,

$$\frac{x-2}{3} = \frac{y+1}{2} = \frac{0-1}{-1}$$

$$\Rightarrow \quad \frac{x-2}{3} = 1 \text{ and } \frac{y+1}{2} = 1$$

$\Rightarrow \quad x = 5$ and $y = 1$

Putting these values in $xy = c^2$, we get

$5 = c^2$ or $c = \pm\sqrt{5}$

3. (1) $4(2) - 2(3) - 1(2) = 0$

Also, point $(-3, 4, -5)$ does not lie on the plane. Therefore, the line is parallel to the plane.

4. (3) The given plane passes through $\vec{a}$ and is parallel to the vectors $\vec{b} - \vec{a}$ and $\vec{c}$. So it is normal to $(\vec{b} - \vec{a}) \times \vec{c}$. Hence, its equation is

$$(\vec{r} - \vec{a}) \cdot ((\vec{b} - \vec{a}) \times \vec{c}) = 0$$

or $\quad \vec{r} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a}) = [\vec{a}\ \vec{b}\ \vec{c}]$

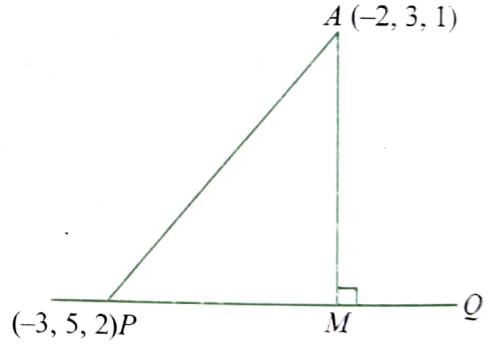
The length of the perpendicular from the origin to this plane is

$$\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}$$

5. (2) Here, $\alpha = \beta = \gamma$

$\because \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \quad \cos\alpha = \dfrac{1}{\sqrt{3}}$



Direction cosines of PQ are $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

PM = Projection of AP on PQ

$$= \left|(-2+3)\frac{1}{\sqrt{3}} + (3-5)\frac{1}{\sqrt{3}} + (1-2)\frac{1}{\sqrt{3}}\right|$$

$$= \frac{2}{\sqrt{3}}$$

and $AP = \sqrt{(-2+3)^2 + (3-5)^2 + (1-2)^2} = \sqrt{6}$

$$AM = \sqrt{(AP)^2 - (PM)^2} = \sqrt{6 - \frac{4}{3}} = \sqrt{\frac{14}{3}}$$

6. (3) Given plane is

$$\vec{r} = (1 + \lambda - \mu)\hat{i} + (2 - \lambda)\hat{j} + (3 - 2\lambda + 2\mu)\hat{k}$$

$$= (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} - 2\hat{k}) + \mu(-\hat{i} + 2\hat{k})$$

which is a plane passing through $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and parallel to the vectors $\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{k}$.

Therefore, it is perpendicular to the vector $\vec{n} = \vec{b} \times \vec{c} = -2\hat{i} - \hat{k}$

Hence, equation of plane is $-2(x-1) + (0)(y-2) - (z-3) = 0$ or $2x + z = 5$

7. (3) $\hat{a} = \pm \dfrac{\vec{n_1} \times \vec{n_2}}{\left|\vec{n_1} \times \vec{n_2}\right|} = \pm \dfrac{2\hat{i} + 5\hat{j} + 3\hat{k}}{\sqrt{38}}$

(where $\vec{n_1}$ and $\vec{n_2}$ are normal to the planes)

8. (1) Equation of the plane containing L_1, $A(x-2) + B(y-1) + C(z+1) = 0$

where $A + 2C = 0$; $A + B - C = 0$

$\Rightarrow$ $A = -2C$, $B = 3C$, $C = C$

$\Rightarrow$ Plane is $-2(x-2) + 3(y-1) + z + 1 = 0$

or $2x - 3y - z - 2 = 0$

Hence, $p = \left|\dfrac{-2}{\sqrt{14}}\right| = \sqrt{\dfrac{2}{7}}$

9. (3) $(1, 2, 3)$ satisfies the plane $x - 2y + z = 0$ and also $(\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$

Since the lines $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{3}$ and $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$ both satisfy $(0, 0, 0)$ and $(1, 2, 3)$, both are same. Given line is obviously parallel to the plane $x - 2y + z = 6$.

10. (1) Vector $\left((3\hat{i} - 2\hat{j} + \hat{k}) \times (4\hat{i} - 3\hat{j} + 4\hat{k})\right)$ is perpendicular to $2\hat{i} - \hat{j} + m\hat{k}$. Thus,

$$\begin{vmatrix} 3 & -2 & 1 \\ 4 & -3 & 4 \\ 2 & -1 & m \end{vmatrix} = 0 \quad \text{or} \quad m = -2$$

11. (1) x intercept is say x_1, i.e.,the plane passes through it. Therefore,

$$x_1 \hat{i} \cdot \vec{n} = q \text{ or } x_1 = \frac{q}{\hat{i} \cdot \vec{n}}$$

12. (2) Let direction ratios of the line be (a, b, c); then

$2a - b + c = 0$

and $a - b - 2c = 0$, i.e., $\dfrac{a}{3} = \dfrac{b}{5} = \dfrac{c}{-1}$

Therefore, direction ratios of the line are $(3, 5, -1)$.

Any point on the given line is $(2 + \lambda, 2 - \lambda, 3 - 2\lambda)$; it lies on the given plane π if

$2(2 + \lambda) - (2 - \lambda) + (3 - 2\lambda) = 4$

or $4 + 2\lambda - 2 + \lambda + 3 - 2\lambda = 4$ or $\lambda = -1$

Therefore, the point of intersection of the line and the plane is $(1, 3, 5)$.

Therefore, equation of the required line is

$$\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-5}{-1}$$

13. (3) Direction ratios of OP are (a, b, c)

Therefore, equation of the plane is

$a(x - a) + b(y - b) + c(z - c) = 0$

i.e., $xa + yb + zc = a^2 + b^2 + c^2$

14. (2) Let a point $(3\lambda + 1, \lambda + 2, 2\lambda + 3)$ of the first line also lies on the second line. Then

$$\frac{3\lambda + 1 - 3}{1} = \frac{\lambda + 2 - 1}{2} = \frac{2\lambda + 3 - 2}{3} \text{ or } \lambda = 1$$

Hence, the point of intersection P of the two lines is $(4, 3, 5)$.

Equation of plane perpendicular to OP, where O is $(0, 0, 0)$, and passing through P is $4x + 3y + 5z = 50$.

15. (2) $1 = \left|(\vec{b} - \vec{a}) \cdot \dfrac{(\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right|$

or $|\vec{b} - \vec{a}| \cos 60° = 1 \Rightarrow AB = 2$

16. (1) $A(1, 1, 1)$, $B(2, 3, 5)$, $C(-1, 0, 2)$ direction ratios of AB are $< 1, 2, 4 >$.

Direction ratios of AC are $< -2, -1, 1 >$.

Therefore, direction ratios of normal to plane ABC are $< 2, -3, 1 >$

As a result, equation of the plane ABC is $2x - 3y + z = 0$.

Let the equation of the required plane be $2x - 3y + z = k$. Then

$$\left|\frac{k}{\sqrt{4 + 9 + 1}}\right| = 2 \quad \text{or} \quad k = \pm 2\sqrt{14}$$

Hence, equation of the required plane is $2x - 3y + z \pm 2\sqrt{14} = 0$

17. (2) Direction cosines of the given line are $\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}$

Hence, the equation of line can be a point in the form

$$\frac{x-2}{1/3}=\frac{y+3}{-2/3}=\frac{z+5}{-2/3}=r$$

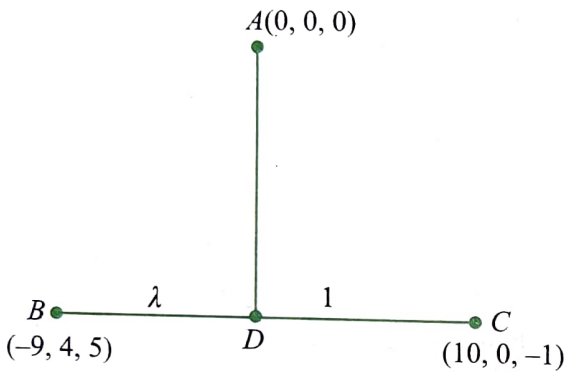
Therefore, any point on the line is

$$\left(2+\frac{r}{3},-3-\frac{2r}{3},-5-\frac{2r}{3}\right), \text{ where } r=\pm 6.$$

Points are $(4,-7,-9)$ and $(0,1,-1)$

18. (4) Let AD be the perpendicular and D be the foot of the perpendicular which divides BC in the ratio $\lambda:1$, then

$$D\left(\frac{10\lambda-9}{\lambda+1},\frac{4}{\lambda+1},\frac{-\lambda+5}{\lambda+1}\right). \qquad \text{(i)}$$



The direction ratios of AD are $\frac{10\lambda-9}{\lambda+1},\frac{4}{\lambda+1}$ and $\frac{-\lambda+5}{\lambda+1}$ and direction ratios of BC are $19,-4$ and -6.

Since $AD \perp BC$, we get

$$19\left(\frac{10\lambda-9}{\lambda+1}\right)-4\left(\frac{4}{\lambda+1}\right)-6\left(\frac{-\lambda+5}{\lambda+1}\right)=0$$

or $\lambda=\frac{31}{28}$

Hence, on putting the value of λ in (i), we get required foot of the perpendicular, i.e., $\left(\frac{58}{59},\frac{112}{59},\frac{109}{59}\right)$.

19. (4) $P_1=P_2=0$, $P_2=P_3=0$ and $P_3=P_1=0$ are lines of intersection of the three planes P_1, P_2 and P_3.

As $\vec{n}_1,\vec{n}_2$ and $\vec{n}_3$ are non-coplanar, planes P_1, P_2 and P_3 will intersect at unique point. So the given lines will pass through a fixed point.

20. (4) Let $A(1,0,-1)$, $B(-1,2,2)$

Direction ratios of segment AB are $<2,-2,-3>$.

$$\therefore \quad \cos\theta=\frac{|2\times1+3(-2)-5(-3)|}{\sqrt{1+9+25}\sqrt{4+4+9}}$$

$$=\frac{11}{\sqrt{17}\sqrt{35}}=\frac{11}{\sqrt{595}}$$

Length of projection $= (AB)\sin\theta$

$$=\sqrt{(2)^2+(2)^2+(3)^2}\times\sqrt{1-\frac{121}{595}}$$

$$=\sqrt{17}\,\frac{\sqrt{474}}{\sqrt{17}\sqrt{35}}=\sqrt{\frac{474}{35}} \text{ units}$$

21. (3) Let the point be A, B, C and D.

The number of planes which have three points on one side and the fourth point on the other side is 4.

The number of planes which have two points on each side of the plane is 3.

Thus, the number of planes is 7.

22. (1) Point A is $(a,b,c)\Rightarrow$ Points P, Q, R are $(a,b,-c)$, $(-a,b,c)$ and $(a,-b,c)$, respectively

$\Rightarrow$ Centroid of triangle PQR is $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

$\Rightarrow$ $G\equiv\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

$\Rightarrow$ A, O, G are collinear

$\Rightarrow$ Area of triangle AOG is zero.

23. (2) Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

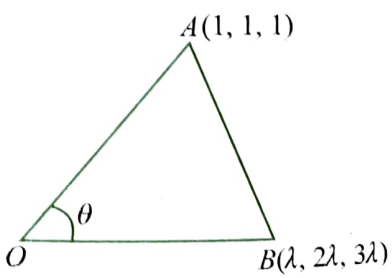
$\Rightarrow$ $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$

$\Rightarrow$ Volume of tetrahedron $OABC=V=\frac{1}{6}(abc)$

Now $(abc)^{1/3}\geq\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\geq 3$ (G.M. $\geq$ H.M.)

or $abc\geq 27\Rightarrow V\geq\frac{9}{2}$

24. (2) Let any point on the second line be $(\lambda,2\lambda,3\lambda)$



$$\cos\theta = \frac{6}{\sqrt{42}},\ \sin\theta = \frac{\sqrt{6}}{\sqrt{42}}$$

$$\Delta_{OAB} = \frac{1}{2}(OA)\ OB \sin\theta$$

$$\Rightarrow \quad \frac{1}{2}\sqrt{3}\ \lambda\ \sqrt{14} \times \frac{\sqrt{6}}{\sqrt{42}} = \sqrt{6}$$

or $\lambda = 2$

So B is $(2, 4, 6)$

(1) Equation of line.

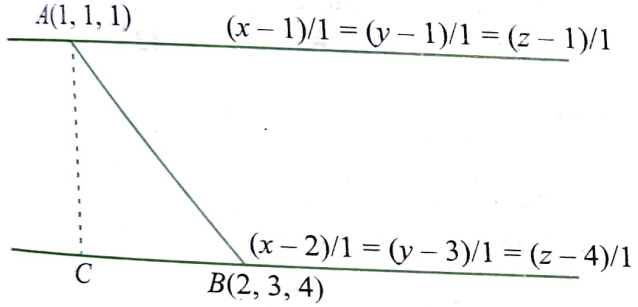
$$x + 2y + z - 1 + \lambda(-x + y - 2z - 2) = 0 \quad \text{(i)}$$

$$x + y - 2 + \mu(x + z - 2) = 0 \quad \text{(ii)}$$

$(0, 0, 1)$ lies on it $\Rightarrow \lambda = 0, \mu = -2$

For point of intersection, $z = 0$ and solve (i) and (ii).

(3) The given lines are parallel.



From the figure, we get

$$BC = \frac{(2-1)1}{\sqrt{3}} + \frac{(3-1)1}{\sqrt{3}} + \frac{(4-1)1}{\sqrt{3}}$$

$$= \frac{1+2+3}{\sqrt{3}} = 2\sqrt{3}$$

$$AB = \sqrt{1+4+9} = \sqrt{14}$$

Shortest distance $= AC = \sqrt{14-12} = \sqrt{2}$

27. (3) Let $Q\ (\vec{q})$ be the foot of altitude drawn from P to the plane $\vec{r}\cdot\vec{n} = 0$. Then

$$\vec{q} - \vec{p} = \lambda\vec{n} \quad \text{or} \quad \vec{q} = \vec{p} + \lambda\vec{n}$$

Also $\vec{q}\cdot\vec{n} = 0 \Rightarrow (\vec{p} + \lambda\vec{n})\cdot\vec{n} = 0$

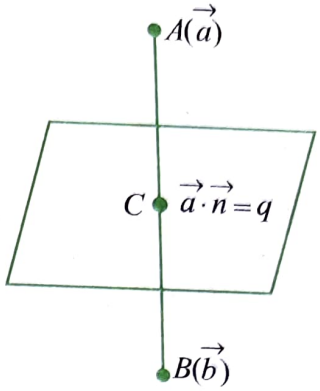
or $\lambda = -\dfrac{\vec{p}\cdot\vec{n}}{|\vec{n}|^2}$ or $\vec{q} - \vec{p} = -\dfrac{(\vec{p}\cdot\vec{n})}{|\vec{n}|^2}\vec{n}$

Thus, required distance

$$= |\vec{q} - \vec{p}| = \frac{|\vec{p}\cdot\vec{n}|}{|\vec{n}|} = |\vec{p}\cdot\hat{n}|$$

28. (2) Given plane is $\vec{r}\cdot\vec{n} = q$ (i)

Let the image of $A\ (\vec{a})$ in the plane be $B\ (\vec{b})$.



Equation of AC is $\vec{r} = \vec{a} + \lambda\vec{n}$

($\because AC$ is normal to the plane) (ii)

Solving (i) and (ii), we get

$$(\vec{a} + \lambda\vec{n})\cdot\vec{n} = q$$

or $$\lambda = \frac{q - \vec{a}\cdot\vec{n}}{|\vec{n}|^2}$$

$$\therefore \quad \overrightarrow{OC} = \vec{a} + \frac{(q - \vec{a}\cdot\vec{n})}{|\vec{n}|^2}\cdot\vec{n}$$

But $$\overrightarrow{OC} = \frac{\vec{a} + \vec{b}}{2}$$

$$\therefore \quad \vec{a} + \frac{(q - \vec{a}\cdot\vec{n})\vec{n}}{|\vec{n}|^2} = \frac{\vec{a} + \vec{b}}{2}$$

or $$\vec{b} = \vec{a} + 2\left(\frac{q - \vec{a}\cdot\vec{n}}{|\vec{n}|^2}\right)\vec{n}$$

29. (3) We must have $\vec{b}\cdot\vec{n} = 0$ (because the line and the plane must be parallel) and $\vec{a}\cdot\vec{n} \neq q$ (as point $\vec{a}$ on the line should not lie on the plane).

30. (3) Here $l = \cos\dfrac{\pi}{4}, m = \cos\dfrac{\pi}{4}$

Let the line make an angle γ with the z-axis. Then

$$l^2 + m^2 + n^2 = 1$$

or $$\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} + \cos^2\gamma = 1$$

or $$\frac{1}{2} + \frac{1}{2} + \cos^2\gamma = 1$$

or $$2\cos^2\gamma = 0 \text{ or } \cos\gamma = 0 \text{ or } \gamma = \frac{\pi}{2}$$

31. (4) Let the plane $\vec{r}\cdot(\vec{i}-2\vec{j}+3\vec{k})=17$ divide the line joining the points $-2\vec{i}+4\vec{j}+7\vec{k}$ and $3\vec{i}-5\vec{j}+8\vec{k}$ in the ratio $t:1$ at point P.

Therefore, point P is

$$\frac{3t-2}{t+1}\vec{i}+\frac{-5t+4}{t+1}\vec{j}+\frac{8t+7}{t+1}\vec{k}$$

This lies on the given plane

$$\therefore \quad \frac{3t-2}{t+1}\cdot(1)+\frac{-5t+4}{t+1}(-2)+\frac{8t+7}{t+1}(3)=17$$

Solving, we get

$$t=\frac{3}{10}$$

32. (4) Let $P(\alpha, \beta, \gamma)$ be the image of the point $Q(-1, 3, 4)$.

Midpoint of PQ lies on $x-2y=0$. Then,

$$\frac{\alpha-1}{2}-2\left(\frac{\beta+3}{2}\right)=0$$

or $\quad \alpha-1-2\beta-6=0$ or $\alpha-2\beta=7 \qquad$ (i)

Also PQ is perpendicular to the plane. Then,

$$\frac{\alpha+1}{1}=\frac{\beta-3}{-2}=\frac{\gamma-4}{0} \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\alpha=\frac{9}{5}, \beta=-\frac{13}{5}, \gamma=4$$

Therefore, image is

$$\left(\frac{9}{5}, -\frac{13}{5}, 4\right)$$

Alternate method:

For image,

$$\frac{\alpha-(-1)}{1}=\frac{\beta-3}{-2}=\frac{\gamma-4}{0}=\frac{-2(-1-2(3))}{(1)^2+(-2)^2}$$

or $\quad \alpha=\frac{9}{5}, \beta=-\frac{13}{5}, \gamma=4$

33. (1) It is obvious that the given line and plane are parallel.

Given point on the line is $A(2, -2, 3)$.

$B(0, 0, 5)$ is a point on the plane. Therefore,

$$\overrightarrow{AB}=(2-0)\hat{i}+(-2-0)\hat{j}+(3-5)\hat{k}$$

Then distance of B from the plane = projection of $\overrightarrow{AB}$ on vector $\hat{i}+5\hat{j}+\hat{k}$

$$p=\left|\frac{\left(2\hat{i}-2\hat{j}-2\hat{k}\right)\cdot\left(\hat{i}+5\hat{j}+\hat{k}\right)}{\sqrt{1+25+1}}\right|$$

$$=\left|\frac{2-10-2}{\sqrt{27}}\right|=\frac{10}{3\sqrt{3}}$$

34. (4) Since line of intersection is perpendicular to both the planes, direction ratios of the line of intersection is

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 1\\ 1 & 3 & 2\end{vmatrix}=3\hat{i}-3\hat{j}+3\hat{k}$$

Hence, $\cos\alpha=\dfrac{3}{\sqrt{9+9+9}}=\dfrac{1}{\sqrt{3}}$

35. (4) Let P be the point $(1, 2, 3)$ and PN be the length of the perpendicular from P on the given line.

Coordinates of point N are $(3\lambda + 6, 2\lambda + 7, -2\lambda + 7)$.

Now PN is perpendicular to the given line or vector $3\vec{i}+2\vec{j}-2\vec{k}$. Thus,

$3(3\lambda + 6 - 1) + 2(2\lambda + 7 - 2) - 2(-2\lambda + 7 - 3) = 0$

or $\quad \lambda=-1$

Then, point N is $(3, 5, 9)$

$\Rightarrow \quad PN=7$

36. (2) The line is $\dfrac{x+1}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{2}$ and the plane is $2x-y+\sqrt{\lambda}z+4=0$.

If θ is the angle between the line and the plane, then $90°-\theta$ is the angle between the line and normal to the plane. Thus,

$$\cos(90°-\theta)=\frac{(1)(2)+(2)(-1)+(2)\left(\sqrt{\lambda}\right)}{\sqrt{1+4+4}\sqrt{4+1+\lambda}}$$

or $\quad \sin\theta=\dfrac{2-2+2\sqrt{\lambda}}{3\sqrt{5+\lambda}}$ or $\dfrac{1}{3}=\dfrac{2\sqrt{\lambda}}{3\sqrt{5+\lambda}}$

or $\quad \sqrt{5+\lambda}=2\sqrt{\lambda}$

or $\quad 5+\lambda=4\lambda$

or $\quad 3\lambda=5$

or $\quad \lambda=\dfrac{5}{3}$

37. (4) The given spheres are

$x^2+y^2+z^2+7x-2y-z-13=0 \qquad$ (i)

and $x^2+y^2+z^2-3x+3y+4z-8=0 \qquad$ (ii)

Subtracting (ii) from (i), we get

$$10x - 5y - 5z - 5 = 0$$

or $\quad 2x - y - z = 1$

38. (3) Plane meets axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Then area of ΔABC

$$= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}|(-a\hat{i} + b\hat{j}) \times (-a\hat{i} + c\hat{k})|$$

$$= \frac{1}{2}\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)}$$

39. (3) Here $\sin^2\beta = 3\sin^2\theta$ (i)

By the question, $\cos^2\theta + \cos^2\theta + \cos^2\beta = 1$ (ii)

$\Rightarrow \quad \cos^2\beta = 1 - 2\cos^2\theta$ (iii)

Adding (i) and (iii), we get

$$1 = 1 + 3\sin^2\theta - 2\cos^2\theta$$

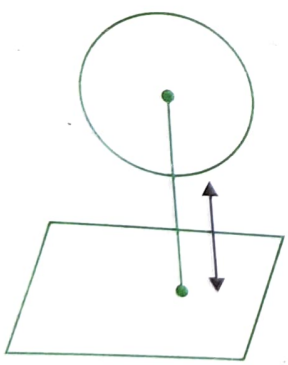
$$= 1 + 3(1 - \cos^2\theta) - 2\cos^2\theta$$

or $\quad 5\cos^2\theta = 3$

or $\quad \cos^2\theta = \frac{3}{5}$

40. (4) The given sphere is

$$x^2 + y^2 + z^2 + 4x - 2y - 6z - 155 = 0$$



Its centre is $(-2, 1, 3)$ and radius

$$= \sqrt{4 + 1 + 9 + 155} = \sqrt{169} = 13$$

Therefore, distance of centre $(-2, 1, 3)$ from the plane $12x + 4y + 3z = 327$ is

$$\frac{|12(-2) + 4(1) + 3(3) - 327|}{\sqrt{144 + 16 + 9}} = 26$$

Hence, the shortest distance is 13.

41. (4) Vector perpendicular to the face OAB is

$$\overrightarrow{OA} \times \overrightarrow{OB} = (\hat{i} + 2\hat{j} + \hat{k}) \times (2\hat{i} + \hat{j} + 3\hat{k})$$

$$= 5\hat{i} - \hat{j} - 3\hat{k}$$

Vector perpendicular to face ABC is

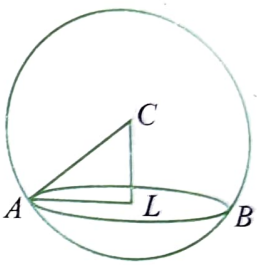
$$\overrightarrow{AB} \times \overrightarrow{AC} = (\hat{i} - \hat{j} + 2\hat{k}) \times (-2\hat{i} - \hat{j} + \hat{k})$$

$$= \hat{i} - 5\hat{j} - 3\hat{k}$$

Since the angle between the face = angle between their normal, therefore

$$\cos\theta = \frac{5 + 5 + 9}{\sqrt{35}\sqrt{35}} = \frac{19}{35} \text{ or } \theta = \cos^{-1}\left(\frac{19}{35}\right)$$

42. (2) Center of the sphere is $(-1, 1, 2)$ and its radius $= \sqrt{1 + 1 + 4 + 19} = 5$



CL, perpendicular distance of C from plane, is

$$\left|\frac{-1 + 2 + 4 + 7}{\sqrt{1 + 4 + 4}}\right| = 4$$

Now $AL^2 = CA^2 - CL^2 = 25 - 16 = 9$

Hence, radius of the circle $= \sqrt{9} = 3$

43. (2) The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ (i)

and $\quad \frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ (ii)

are coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_1 & n_2 \end{vmatrix} = 0$

or $\quad \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$

or $\quad k^2 + 3k = 0$

$\Rightarrow \quad k = 0$ or -3

44. (1) Given lines are

$$\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1} = r_1 \quad \text{(say)}$$

and $\quad \frac{x+3}{-36} = \frac{y-3}{2} = \frac{z-6}{4} = r_2 \quad \text{(say)}$

$\therefore \quad x = 3r_1 + 5 = -36r_2 - 3,$

$y = -r_1 + 7 = 3 + 2r_2$

and $\quad z = r_1 - 2 = 4r_2 + 6$

On solving, we get

$$x = 21, y = \frac{5}{3}, z = \frac{10}{3}$$

45. (3) The planes are $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and $\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=1$

Since the perpendicular distance of the origin on the planes is same, therefore

$$\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| = \left|\frac{-1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}\right|$$

or $$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2}=0$$

46. (1) The required plane is $\begin{vmatrix} x-3 & y-6 & z-4 \\ 3-3 & 2-6 & 0-4 \\ 1 & 5 & 4 \end{vmatrix}=0$

or $$\begin{vmatrix} x-3 & y-z-2 & z-4 \\ 0 & 0 & -4 \\ 1 & 1 & 4 \end{vmatrix}=0$$

(Operating $C_2 \to C_2 - C_3$)

or $4(x-3-y+z+2)=0$

or $x-y+z=1$

47. (2) Any plane through (1, 0, 0) is

$$a(x-1)+by+cz=0 \qquad \text{(i)}$$

It passes through (0, 1, 0). Therefore,

$$a(0-1)+b(1)+c(0)=0 \text{ or } -a+b=0 \qquad \text{(ii)}$$

(i) makes an angle of $\frac{\pi}{4}$ with $x+y=3$, therefore

$$\cos\frac{\pi}{4}=\frac{a(1)+b(1)+c(0)}{\sqrt{a^2+b^2+c^2}\sqrt{1+1+0}}$$

or $$\frac{1}{\sqrt{2}}=\frac{a+b}{\sqrt{2}\sqrt{a^2+b^2+c^2}}$$

or $$a+b=\sqrt{a^2+b^2+c^2}$$

Squaring, we get

$$a^2+b^2+2ab=a^2+b^2+c^2$$

or $2ab=c^2$ or $2a^2=c^2$ [using (ii)]

or $c=\sqrt{2}a$

Hence, $a:b:c = a:a:\sqrt{2}a$

$= 1:1:\sqrt{2}$

48. (2) The equation of the line through the centre $\hat{j}+2\hat{k}$ and normal to the given plane is

$$\vec{r}=\hat{j}+2\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k}) \qquad \text{(i)}$$

This meets the plane for which

$$[\hat{j}+2\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})]\cdot(\hat{i}+2\hat{j}+2\hat{k})=15$$

or $6+9\lambda=15$ or $\lambda=1$

Putting in (i), we get

$$\vec{r}=\hat{j}+2\hat{k}+(\hat{i}+2\hat{j}+2\hat{k})=\hat{i}+3\hat{j}+4\hat{k}$$

Hence, centre is (1, 3, 4).

49. (3) Equations of the planes through $y=mx, z=c$ and $y=-mx, z=-c$ are respectively,

$$(y-mx)+\lambda_1(z-c)=0 \qquad \text{(i)}$$

and $$(y+mx)+\lambda_2(z+c)=0 \qquad \text{(ii)}$$

It meets at x-axis, i.e., $y=0=z$.

$\therefore \quad \lambda_2=\lambda_1$

From (i) and (ii), $\frac{y-mx}{z-c}=\frac{y+mx}{z+c}$

$\therefore \quad cy=mzx$

50. (3) Let $Q(\vec{q})$ be the foot of altitude drawn from $P(\vec{p})$ to the line $\vec{r}=\vec{a}+\lambda\vec{b}$,

$\Rightarrow \quad (\vec{q}-\vec{p})\cdot\vec{b}=0$ and $\vec{q}=\vec{a}+\lambda\vec{b}$

$\Rightarrow \quad (\vec{a}+\lambda\vec{b}-\vec{p})\cdot\vec{b}=0$

or $$(\vec{a}-\vec{p})\cdot\vec{b}+\lambda|\vec{b}|^2=0$$

or $$\lambda=\frac{(\vec{p}-\vec{a})\cdot\vec{b}}{|\vec{b}|^2}$$

$\Rightarrow \quad \vec{q}-\vec{p}=\vec{a}+\frac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}-\vec{p}$

$\Rightarrow \quad |\vec{q}-\vec{p}|=\left|(\vec{a}-\vec{p})+\frac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$

51. (2) Coordinates of L and M are $(0, b, c)$ and $(a, 0, c)$, respectively. Therefore, the equation of the plane passing through (0, 0, 0), $(0, b, c)$ and $(a, 0, c)$ is

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 0 & b & c \\ a & 0 & c \end{vmatrix}=0 \text{ or } \frac{x}{a}+\frac{y}{b}-\frac{z}{c}=0$$

52. (3) We must have $\vec{b}\cdot\vec{n}=0$ and $\vec{a}\cdot\vec{n}=q$.

53. (2) We have $\vec{s}-\vec{p}=\lambda\vec{n}$ and $\vec{s}\cdot\vec{n}=q$. Thus,

$$(\lambda\vec{n}+\vec{p})\cdot\vec{n}=q$$

or $\lambda = \dfrac{q - \vec{p}\cdot\vec{n}}{|\vec{n}|^2}$

$\Rightarrow \quad \vec{s} = \vec{p} + \dfrac{(q - \vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

54. **(4)** Line of intersection of $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=0$ and $\vec{r}\cdot(3\hat{i}+3\hat{j}+\hat{k})=0$ will be parallel to $(3\hat{i}+3\hat{j}+\hat{k})\times(\hat{i}+2\hat{j}+3\hat{k})$, i.e., $7\hat{i}-8\hat{j}+3\hat{k}$.

If the required angle is θ, then

$$\cos\theta = \frac{7}{\sqrt{49+64+9}} = \frac{7}{\sqrt{122}}$$

55. **(3)** Given one vertex $A(7, 2, 4)$ and line

$$\frac{x+6}{5} = \frac{y+10}{3} = \frac{z+14}{8}$$

General point on above line $B \equiv (5\lambda - 6, 3\lambda - 10, 8\lambda - 14)$

Direction ratios of line AB are $<5\lambda - 13, 3\lambda - 12, 8\lambda - 18>$

Direction ratios of line BC are $<5, 3, 8>$

Since angle between AB and BC is $\pi/4$, we have

$$\cos\frac{\pi}{4} = \frac{(5\lambda-3)5 + 3(3\lambda-12) + 8(8\lambda-18)}{\sqrt{5^2+3^2+8^2}\cdot\sqrt{(5\lambda-13)^2+(3\lambda-12)^2+(8\lambda-18)^2}}$$

Squaring and solving, we have $\lambda = 3, 2$

Hence, equation of lines are $\dfrac{x-7}{2} = \dfrac{y-2}{-3} = \dfrac{z-4}{6}$

and $\dfrac{x-7}{3} = \dfrac{y-2}{6} = \dfrac{z-4}{2}$

56. **(1)** $\vec{r}\cdot\vec{n_1} + \lambda\vec{r}\cdot\vec{n_2} = q_1 + \lambda q_2$ (i)

where λ is a parameter.

So, $\vec{n_1} + \lambda\vec{n_2}$ is normal to plane (i). Now, any plane parallel to the line of intersection of the planes $\vec{r}\cdot\vec{n_3} = q_3$ and $\vec{r}\cdot\vec{n_4} = q_4$ is of the form $\vec{r}\cdot(\vec{n_3}\times\vec{n_4}) = d$. Hence, we must have

$$[\vec{n_1} + \lambda\vec{n_2}]\cdot[\vec{n_3}\times\vec{n_4}] = 0$$

or $\quad [\vec{n_1}\,\vec{n_3}\,\vec{n_4}] + \lambda[\vec{n_2}\,\vec{n_3}\,\vec{n_4}] = 0$

or $\quad \lambda = \dfrac{-[\vec{n_1}\,\vec{n_3}\,\vec{n_4}]}{[\vec{n_2}\,\vec{n_3}\,\vec{n_4}]}$

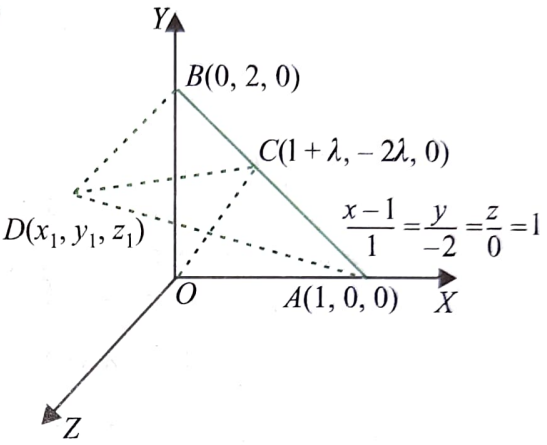
On putting this value in Eq. (i), we have the equation of the required plane as

$$\vec{r}\cdot\vec{n_1} - q_1 = \frac{[\vec{n_1}\,\vec{n_3}\,\vec{n_4}]}{[\vec{n_2}\,\vec{n_3}\,\vec{n_4}]}(\vec{r}\cdot\vec{n_2} - q_2)$$

or $\quad [\vec{n_2}\,\vec{n_3}\,\vec{n_4}](\vec{r}\cdot\vec{n_1} - q_1)$

$$= [\vec{n_1}\,\vec{n_3}\,\vec{n_4}](\vec{r}\cdot\vec{n_2} - q_2)$$

57. **(3)**



Equation of line AB is $\dfrac{x-1}{1} = \dfrac{y}{-2} = \dfrac{z}{0} = \lambda$

Now $AB \perp OC \Rightarrow 1(\lambda + 1) + (-2\lambda)(-2) = 0$

or $\quad 5\lambda = -1$ or $\lambda = -\dfrac{1}{5}$

$\Rightarrow \quad C$ is $\left(\dfrac{4}{5}, \dfrac{2}{5}, 0\right)$. Now

$$x_1^2 + (y_1 - 2)^2 + z_1^2 = 4 \qquad \text{(i)}$$

and $\quad (x_1 - 1)^2 + y_1^2 + z_1^2 = 1 \qquad$ (ii)

Now $OC \perp CD$

$\Rightarrow \quad \left(x_1 - \dfrac{4}{5}\right)\dfrac{4}{5} + \left(y_1 - \dfrac{2}{5}\right)\dfrac{2}{5} + (z_1 - 0)\,0 = 0$ (iii)

From (i) and (ii), we get

$$-4y_1 + 2x_1 = 0 \text{ or } x_1 = 2y_1$$

From (iii), putting $x_1 = 2y_1 \Rightarrow 2y_1 = \dfrac{4}{5}$ or $y_1 = \dfrac{2}{5}$

$\Rightarrow x_1 = \dfrac{4}{5}$. Putting this value of x_1 and y_1 in (i), we get

$$z_1 = \pm\frac{2}{\sqrt{5}}$$

58. **(2)** Let $\vec{r}\times\vec{a} = \vec{b}\times\vec{a}$

$\Rightarrow \quad (\vec{r} - \vec{b})\times\vec{a} = \vec{0} \Rightarrow \vec{r} = \vec{b} + t\vec{a}$

Similarly, other line $\vec{r} = \vec{a} + k\vec{b}$, where t and k are scalars.

Now $\vec{a} + k\vec{b} = \vec{b} + t\vec{a}$

$\Rightarrow \quad t = 1, k = 1$

(equating the coefficients of $\vec{a}$ and $\vec{b}$)

$\therefore \quad \vec{r} = \vec{a} + \vec{b} = \hat{i} + \hat{j} + 2\hat{i} - \hat{k} = 3\hat{i} + \hat{j} - \hat{k}$

i.e., $(3, 1, -1)$

59. (1) Let the point P be (x, y, z), then the vector $x\hat{i} + y\hat{j} + z\hat{k}$ will lie on the line. Thus,

$(x-1)\hat{i} + (y-1)\hat{j} + (z-1)\hat{k} = -\lambda\hat{i} + \lambda\hat{j} - \lambda\hat{k}$

$\Rightarrow \quad x = 1 - \lambda, y = 1 + \lambda$ and $z = 1 - \lambda$

Now point P is nearest to the origin

$\Rightarrow \quad D = (1-\lambda)^2 + (1+\lambda)^2 + (1-\lambda)^2$

or $\quad \dfrac{dD}{d\lambda} = -4(1-\lambda) + 2(1+\lambda) = 0 \Rightarrow \lambda = \dfrac{1}{3}$

Hence, the point is $\left(\dfrac{2}{3}, \dfrac{4}{3}, \dfrac{2}{3}\right)$

60. (2) Let P be the point and it divides the line segment in the ratio $\lambda : 1$. Then,

$$\overrightarrow{OP} = \vec{r} = \frac{-3\lambda + 2}{\lambda + 1}\hat{i} + \frac{5\lambda - 4}{\lambda + 1}\hat{j} + \frac{-8\lambda - 7}{\lambda + 1}\hat{k}$$

It satisfies $\vec{r} \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 13$. So,

$$\frac{-3\lambda + 2}{\lambda + 1} - 2\frac{5\lambda - 4}{\lambda + 1} + 3\frac{-8\lambda - 7}{\lambda + 1} = 13$$

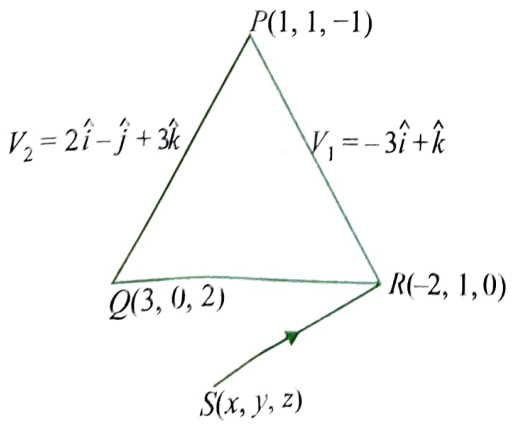
or $\quad -3\lambda + 2 - 2(5\lambda - 4) + 3(-8\lambda - 7) = 13(\lambda + 1)$

or $\quad -37\lambda - 11 = 13\lambda + 13$ or $50\lambda = -24$ or $\lambda = -\dfrac{12}{25}$

61. (4) $\vec{V}_1, \vec{V}_2, \overrightarrow{PS}$ are in the same plane

$\therefore \quad (2\hat{i} - \hat{j} + 3\hat{k}) \times (-3\hat{i} + \hat{k}) \cdot$

$((x+2)\hat{i} + (y-1)\hat{j} + z\hat{k}) = 0$



62. (1) $\overrightarrow{AB} = \vec{\beta} - \vec{\alpha} = -2\hat{i} - 3\hat{j} - 6\hat{k}$

Equation of the plane passing through B and perpendicular to AB is

$$(\vec{r} - \overrightarrow{OB}) \cdot \overrightarrow{AB} = 0$$

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) + 28 = 0$$

Hence, the required distance from

$$\vec{r} = -\hat{i} + \hat{j} + \hat{k}$$

$$= \left|\frac{(-\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) + 28}{|2\hat{i} + 3\hat{j} + 6\hat{k}|}\right|$$

$$= \left|\frac{-2 + 3 + 6 + 28}{7}\right|$$

$= 5$ units

63. (1) Both the lines pass through the origin. Line L_1 is parallel to the vector $\vec{V}_1$

$$\vec{V}_1 = (\cos\theta + \sqrt{3})\hat{i} + (\sqrt{2}\sin\theta)\hat{j} + (\cos\theta - \sqrt{3})\hat{k}$$

and L_2 is parallel to the vector $\vec{V}_2$

$$\vec{V}_2 = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\therefore \quad \cos\alpha = \frac{\vec{V}_1 \cdot \vec{V}_2}{|\vec{V}_1||\vec{V}_2|}$$

$$= \frac{a(\cos\theta + \sqrt{3}) + (b\sqrt{2})\sin\theta + c(\cos\theta - \sqrt{3})}{\sqrt{a^2 + b^2 + c^2}\sqrt{(\cos\theta + \sqrt{3})^2 + 2\sin^2\theta + (\cos\theta - \sqrt{3})^2}}$$

$$= \frac{(a+c)\cos\theta + b\sqrt{2}\sin\theta + (a-c)\sqrt{3}}{\sqrt{a^2 + b^2 + c^2}\sqrt{2+6}}$$

For $\cos\alpha$ to be independent of θ, we get

$a + c = 0$ and $b = 0$

$$\therefore \quad \cos\alpha = \frac{2a\sqrt{3}}{a\sqrt{2}\,2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

or $\quad \alpha = \dfrac{\pi}{6}$

64. (4) Given lines are $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + l(3\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + m(-3\hat{i} + 2\hat{j} + 4\hat{k})$

Required shortest distance

$$= \frac{|(6\hat{i} + 15\hat{j} - 3\hat{k}) \cdot ((3\hat{i} - \hat{j} + \hat{k}) \times (-3\hat{i} + 2\hat{j} + 4\hat{k}))|}{|(3\hat{i} - \hat{j} + \hat{k}) \times (-3\hat{i} + 2\hat{j} + 4\hat{k})|}$$

$$= \frac{|(6\hat{i}+15\hat{j}-3\hat{k})\cdot(-6\hat{i}-15\hat{j}+3\hat{k})|}{|-6\hat{i}-15\hat{j}+3\hat{k}|}$$

$$= \frac{36+225+9}{\sqrt{36+225+9}} \quad \frac{270}{\sqrt{270}} = \sqrt{270} = 3\sqrt{30}$$

65. (2) The required line passes through the point $\hat{i}+3\hat{j}+2\hat{k}$ and is perpendicular to the lines $\vec{r} = (\hat{i}+2\hat{j}-\hat{k}) + \lambda(2\hat{i}+\hat{j}+\hat{k})$ and $\vec{r} = (2\hat{i}+6\hat{j}+\hat{k}) + \mu(\hat{i}+2\hat{j}+3\hat{k})$; therefore, it is parallel to the vector $\vec{b} = (2\hat{i}+\hat{j}+\hat{k}) \times (\hat{i}+2\hat{j}+3\hat{k}) = (\hat{i}-5\hat{j}+3\hat{k})$

Hence, the equation of the required line is

$$\vec{r} = (\hat{i}+3\hat{j}+2\hat{k}) + \lambda(\hat{i}-5\hat{j}+3\hat{k})$$

66. (4) Here, the required plane is

$$a(x-4) + b(y-3) + c(z-2) = 0$$

Also $a + b + 2c = 0$ and $a - 4b + 5c = 0$

Solving, we have

$$\frac{a}{5+8} = \frac{b}{2-5} = \frac{c}{-4-1} = k$$

$$\frac{a}{13} = \frac{b}{-3} = \frac{c}{-5} = k$$

Therefore, the required equation of plane is

$$-13x + 3y + 5z + 33 = 0$$

67. (2) Plane passing through the line of intersection of planes $4y + 6z = 5$ and $2x + 3y + 5z = 5$ is

$$(4y + 6z - 5) + \lambda(2x + 3y + 5z - 5) = 0$$

or $\quad 2\lambda x + (3\lambda + 4)y + (5\lambda + 6)z - 5\lambda - 5 = 0$

Clearly, for $\lambda = -3$, we get the plane $6x + 5y + 9z = 10$.

Hence, the given three planes have common line of intersection.

68. (3) The equation of a plane through the line of intersection of the planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ is

$$(ax + by + cz + d) + \lambda(a'x + b'y + c'z + d') = 0$$

or $\quad x(a + \lambda a') + y(b + \lambda b') + z(c + \lambda c') + d + \lambda d' = 0 \quad$ (i)

This is parallel to the x-axis, i.e., $y = 0$, $z = 0$. Therefore,

$$1(a + \lambda a') + 0(b + \lambda b') + 0(c + \lambda c') = 0$$

or $\quad \lambda = -\frac{a}{a'}$

Putting the value of λ in (i), the required plane is

$$y(a'b - ab') + z(a'c - ac') + a'd - ad' = 0$$

or $\quad (ab' - a'b)y + (ac' - a'c)z + ad' - a'd = 0$

69. (2) Any plane through (2, 2, 1) is

$$a(x-2) + b(y-2) + c(z-1) = 0 \quad \text{(i)}$$

It passes through (9, 3, 6) if $7a + b + 5c = 0$. (ii)

Also (i) is perpendicular to $2x + 6y + 6z - 1 = 0$, we have

$$2a + 6b + 6c = 0$$

$$\therefore \quad a + 3b + 3c = 0 \quad \text{(iii)}$$

$$\therefore \quad \frac{a}{-12} = \frac{b}{-16} = \frac{c}{20} \text{ or } \frac{a}{3} = \frac{b}{4} = \frac{c}{-5}$$

[from (ii) and (iii)]

Therefore, the required plane is $3(x-2) + 4(y-2) - 5(z-1) = 0$ or $3x + 4y - 5z - 9 = 0$.

70. (1) Since line is parallel to the plane, vector $2\vec{i} + 3\vec{j} + \lambda\vec{k}$ is perpendicular to the normal to the plane $2\vec{i} + 3\vec{j} + 4\vec{k}$

$$\Rightarrow \quad 2 \times 2 + 3 \times 3 + 4\lambda = 0$$

or $\quad \lambda = -\frac{13}{4}$

71. (1) Any plane through the given planes is

$$x + 2y + 3z - 4 + \lambda(4x + 3y + 2z + 1) = 0$$

It passes through (0, 0, 0). Therefore,

$$-4 + \lambda = 0$$

or $\quad \lambda = 4$

Therefore, the required plane is $x + 2y + 3z + 4(4x + 3y + 2z) = 0$ or $17x + 14y + 11z = 0$.

72. (1) The equation of the plane through the line of intersection of the planes $4x + 7y + 4z + 81 = 0$ and $5x + 3y + 10z = 25$ is

$$(4x + 7y + 4z + 81) + \lambda(5x + 3y + 10z - 25) = 0$$

or $\quad (4 + 5\lambda)x + (7 + 3\lambda)y + (4 + 10\lambda)z + 81 - 25\lambda = 0 \quad$ (i)

which is perpendicular to $4x + 7y + 4z + 81 = 0$

$$\Rightarrow \quad 4(4 + 5\lambda) + 7(7 + 3\lambda) + 4(4 + 10\lambda) = 0$$

or $\quad 81\lambda + 81 = 0$

or $\quad \lambda = -1$

Hence, the plane is $x - 4y + 6z = 106$

73. (2) The equation of a plane through the line of intersection of the planes $\vec{r}\cdot\vec{a} = \lambda$ and $\vec{r}\cdot\vec{b} = \mu$ is

$$(\vec{r}\cdot\vec{a} - \lambda) + k(\vec{r}\cdot\vec{b} - \mu) = 0$$

or $\quad \vec{r}\cdot(\vec{a} + k\vec{b}) = \lambda + k\mu \quad$ (i)

This passes through the origin, therefore

$$\vec{0}\,(\vec{a}+k\vec{b}) = \lambda+\mu k \quad \text{or } k = \frac{-\lambda}{\mu}$$

Putting the value of k in (i), we get the equation of the required plane as

$$\vec{r}\cdot(\mu\vec{a}-\lambda\vec{b}) = 0 \text{ or } \vec{r}\cdot(\lambda\vec{b}-\mu\vec{a}) = 0$$

74. (2) The lines $\vec{r} = \vec{a}+\lambda(\vec{b}\times\vec{c})$ and $\vec{r} = \vec{b}+\mu(\vec{c}\times\vec{a})$ pass through points $\vec{a}$ and $\vec{b}$, respectively, and are parallel to the vectors $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$, respectively. Therefore, they intersect if $\vec{a}-\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are coplanar and so

$$(\vec{a}-\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\} = 0$$

or $$(\vec{a}-\vec{b})\cdot([\vec{b}\,\vec{c}\,\vec{a}]\,\vec{c}-[\vec{b}\,\vec{c}\,\vec{c}]\,\vec{a}) = 0$$

or $$((\vec{a}-\vec{b})\cdot\vec{c})[\vec{b}\,\vec{c}\,\vec{a}] = 0$$

or $$\vec{a}\cdot\vec{c}-\vec{b}\cdot\vec{c} = 0 \text{ or } \vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$$

75. (1) Equation of the plane through (–1, 0, 1) is

$$a(x+1)+b(y-0)+c(z-1) = 0 \qquad \text{(i)}$$

which is parallel to the given line and perpendicular to the given plane

$$-a+2b+3c = 0 \qquad \text{(ii)}$$

and $$a-2b+c = 0 \qquad \text{(iii)}$$

From Eqs. (ii) and (iii), we get

$$c = 0,\ a = 2b$$

From Eq. (i), $2b(x+1)+by = 0$

$\Rightarrow$ $2x+y+2 = 0$

76. (2) Eliminating n, we get

$$\lambda(l+m)^2+lm = 0$$

$$\Rightarrow \quad \frac{\lambda l^2}{m^2}+(2\lambda+1)\frac{l}{m}+\lambda = 0$$

$$\Rightarrow \quad \frac{l_1l_2}{m_1m_2} = 1 \text{ (product of roots } \frac{l_1}{m_1} \text{ and } \frac{l_2}{m_2}\text{)}$$

where l_1/m_1 and l_2/m_2 are the roots of this equation, further eliminating m, we get

$$\lambda l^2-ln-n^2 = 0$$

$$\Rightarrow \quad \frac{l_1l_2}{n_1n_2} = -\frac{1}{\lambda}$$

Since the lines with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular, we have

$$l_1l_2+m_1m_2+n_1n_2 = 0$$

or $$1+1-\lambda = 0$$

or $$\lambda = 2$$

77. (1) Direction ratios of the line joining points $P(1, 2, 3)$ and $Q(-3, 4, 5)$ are $-4, 2, 2$ which are direction ratios of the normal to the plane.

Then, equation of plane is $-4x+2y+2z = k$.

Also this plane passes through the midpoint of $PQ(-1, 3, 4)$. Thus,

$$-4(-1)+2(3)+2(4) = k$$

or $$k = 18$$

Hence, equation of plane is $2x-y-z = -9$

Then, intercepts are $(-9/2)$, 9 and 9

78. (3) $3l+m+5n = 0$ (i)

$$6mn-2nl+5ml = 0 \qquad \text{(ii)}$$

Substituting the value of n from (i) in (ii), we get

$$6l^2+9lm-6m^2 = 0$$

or $$6\left(\frac{l}{m}\right)^2+9\left(\frac{l}{m}\right)-6 = 0$$

$$\therefore \quad \frac{l_1}{m_1} = \frac{1}{2} \text{ and } \frac{l_2}{m_2} = -2$$

From Eq. (i), we get

$$\frac{l_1}{n_1} = -1 \text{ and } \frac{l_2}{n_2} = -2$$

$$\therefore \quad \frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{-1} = \sqrt{\frac{l_1^2+m_1^2+n_1^2}{1+4+1}} = \frac{1}{\sqrt{6}}$$

and $$\frac{l_2}{2} = \frac{m_2}{-1} = \frac{n_2}{-1} = \frac{\sqrt{l_2^2+m_2^2+n_2^2}}{\sqrt{4+1+1}} = \frac{1}{\sqrt{6}}$$

If θ be the angle between the lines, then

$$\cos\theta = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{2}{\sqrt{6}}\right)+\left(\frac{2}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{6}}\right)$$

$$+\left(-\frac{1}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{6}}\right) = \frac{1}{6}$$

or $$\theta = \cos^{-1}\left(\frac{1}{6}\right)$$

79. (2) Let the equation of the sphere be $x^2+y^2+z^2-ax-by-cz = 0$. This meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let (α, β, γ) be the coordinates of the centroid of the tetrahedron $OABC$. Then

$$\frac{a}{4} = \alpha,\ \frac{b}{4} = \beta,\ \frac{c}{4} = \gamma$$

or $a = 4\alpha, b = 4\beta, c = 4\gamma$

Now, radius of the sphere $= 2k$

$\Rightarrow \quad \frac{1}{2}\sqrt{a^2+b^2+c^2} = 2k$ or $a^2 + b^2 + c^2 = 16k^2$

or $16(\alpha^2 + \beta^2 + \gamma^2) = 16k^2$

Hence, the locus of (α, β, γ) is $(x^2 + y^2 + z^2) = k^2$

80. (1) Let the foot of the perpendicular from the origin on the given plane be $P(\alpha, \beta, \gamma)$. Since the plane passes through $A(a, b, c)$,

$$AP \perp OP \Rightarrow \overrightarrow{AP} \cdot \overrightarrow{OP} = 0$$

$$\Rightarrow \quad [(\alpha - a)\hat{i} + (\beta - b)\hat{j} + (\gamma - c)\hat{k}] \cdot (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) = 0$$

or $\alpha(\alpha - a) + \beta(\beta - b) + \gamma(\gamma - c) = 0$

Hence, the locus of (α, β, γ) is

$$x(x - a) + y(y - b) + z(z - c) = 0$$
$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

which is a sphere of radius $\frac{1}{2}\sqrt{a^2+b^2+c^2}$

81. (3) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = 0$

82. (1) Foot of the perpendicular from point $A(\vec{a})$ on the plane $\vec{r} \cdot \vec{n} = d$ is $\vec{a} + \frac{\left(d - \vec{a} \cdot \vec{n}\right)}{|\vec{n}|^2}\vec{n}$

Therefore, equation of the line parallel to $\vec{r} = \vec{a} + \lambda\vec{b}$ in the plane $\vec{r} \cdot \vec{n} = d$ is given by

$$\vec{r} = \vec{a} + \frac{\left(d - \vec{a} \cdot \vec{n}\right)}{|\vec{n}|^2}\vec{n} + \lambda\vec{b}$$

83. (1) The plane is perpendicular to the line

$$\frac{x-a}{\cos\theta} = \frac{y+2}{\sin\theta} = \frac{z-3}{0}.$$

Hence, the direction ratios of the normal of the plane are $\cos\theta$, $\sin\theta$ and 0. (i)

Now, the required plane passes through the z-axis.

Hence, the point $(0, 0, 0)$ lies on the plane.

From Eqs. (i) and (ii), we get equation of the plane as

$$\cos\theta(x - 0) + \sin\theta(y - 0) + 0(z - 0) = 0$$
$$\cos\theta\, x + \sin\theta\, y = 0$$
$$x + y\tan\theta = 0$$

84. (1) The given line makes angles of $\pi/4$, $\pi/4$ and $\pi/2$ with the x-, y- and z-axes, respectively.

Hence, direction cosines of the given line are

$\cos(\pi/4), \cos(\pi/4)$ and $\cos(\pi/2)$,

or $(1/\sqrt{2}), (1/\sqrt{2})$ and 0.

85. (1) We must have $(3 + 4a - 12 + 13)(-9 - 12a + 13) < 0$.

or $(a + 1)(12a - 4) > 0$

or $a < -1$ or $a > 1/3$

86. (3) Plane meets axes at $A(2, 0, 0), B(0, 3, 0)$ and $C(0, 0, 6)$.

Then area of ΔABC is

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}|(-2\hat{i} + 3\hat{j}) \times (-2\hat{i} + 6\hat{j})|$$

$$= 3\sqrt{14} \text{ sq units}$$

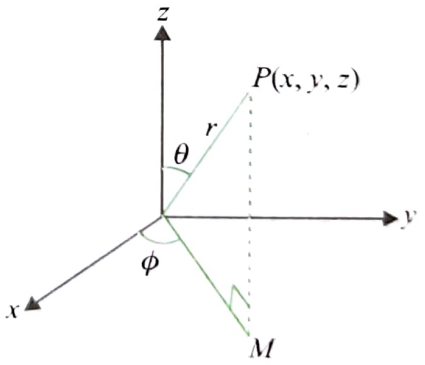
Multiple Correct Answers Type

1. (2), (3), (4)

If P be (x, y, z), then from the figure,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$

$1 = r\sin\theta\cos\phi$, $2 = r\sin\theta\sin\phi$ and $3 = r\cos\theta$



$$\Rightarrow \quad 1^2 + 2^2 + 3^2 = r^2 \text{ or } r = \pm\sqrt{14}$$

$$\therefore \quad \sin\theta\cos\phi = \frac{1}{\sqrt{14}}, \sin\theta\sin\phi = \frac{2}{\sqrt{14}} \text{ and } \cos\theta = \frac{3}{\sqrt{14}}$$

(neglecting negative sign as θ and ϕ are acute)

$$\frac{\sin\theta\sin\phi}{\sin\theta\cos\phi} = \frac{2}{1} \text{ or } \tan\phi = 2$$

Also, $\tan\theta = \sqrt{5}/3$

2. (1), (3)

Plane P_1 contains the line $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$, hence contains the point $\hat{i} + \hat{j} + \hat{k}$ and is normal to vector $(\hat{i} + \hat{j})$.

Hence, equation of plane is

$$(\vec{r} - (\hat{i} + \hat{j} + \hat{k})) \cdot (\hat{i} + \hat{j}) = 0$$

or $x + y = 2$

Plane P_2 contains the line

$$\vec{r} = \hat{i} + \hat{j} + \hat{k} + \quad (\hat{i} - \hat{j} - \hat{k}) \text{ and point } \hat{j}$$

Hence, equation of plane is

$$\begin{vmatrix} x-0 & y-1 & z-0 \\ 1-0 & 1-1 & 1-0 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

or $x + 2y - z = 2$

If θ is the acute angle between P_1 and P_2, then

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \left| \frac{(\hat{i} + \hat{j}) \cdot (\hat{i} + 2\hat{j} - \hat{k})}{\sqrt{2} \cdot \sqrt{6}} \right|$$

$$= \frac{3}{\sqrt{2} \cdot \sqrt{6}} = \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

As L is contained in $P_2 \Rightarrow \theta = 0$

3. (1), (2)

$\vec{r} \cdot \vec{n_1} = \vec{q_1}$ and $\vec{r} \cdot \vec{n_2} = \vec{q_2}$, $\vec{r} \cdot \vec{n_3} = \vec{q_3}$ intersect in a line if $[\vec{n_1} \ \vec{n_2} \ \vec{n_3}] = 0$. So,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2a & 1 \\ a & a^2 & 1 \end{vmatrix} = 0$$

or $2a - a^2 - 1 + a + a^2 - 2a^2 = 0$

or $2a^2 - 3a + 1 = 0$

or $a = 1/2, 1$

4. (1), (2)

Let the coordinates of the point(s) be a, b and c. Therefore, the equation of the line passing through (a, b, c) and whose direction ratios are 1, –5 and –2 is

$$\frac{x-a}{1} = \frac{y-b}{-5} = \frac{z-c}{-2} \qquad \text{(i)}$$

Line (i) intersects the line,

$$\frac{x}{1} = \frac{y+5}{1} = \frac{z+1}{1} \qquad \text{(ii)}$$

Therefore, these are coplanar.

$$\begin{vmatrix} 1 & -5 & -2 \\ 1 & 1 & 1 \\ a & b+5 & c+1 \end{vmatrix} = 0$$

or $a + b - 2c + 3 = 0$

Also, by using same procedure with the second equation, we get the condition

$$11a + 15b - 32c + 55 = 0$$

5. (1), (4)

The equation of the plane passing through the intersection of the planes $2x - y = 0$ and $3z - y = 0$ is

$$2x - y + \lambda(3z - y) = 0 \qquad \text{(i)}$$

or $2x - y(\lambda + 1) + 3\lambda z = 0$

Plane (i) is perpendicular to $4x + 5y - 3z = 8$. Therefore,

$$4 \times 2 - 5(\lambda + 1) - 9\lambda = 0$$

or $8 - 5\lambda - 5 - 9\lambda = 0$

or $3 - 14\lambda = 0$

or $\lambda = 3/14$

$\therefore \quad 2x - y + \frac{3}{14}(3z - y) = 0$

$$28x - 17y + 9z = 0$$

6. (2), (3), (4)

$$x + y + z - 1 = 0$$

$$4x + y - 2z + 2 = 0$$

Therefore, the line is along the vector

$$(\hat{i} + \hat{j} + \hat{k}) \times (4\hat{i} + \hat{j} - 2\hat{k}) = 3\hat{i} - 6\hat{j} + 3\hat{k}$$

Let $z = k$. Then $x = k - 1$ and $y = 2 - 2k$

Therefore, $(k - 1, 2 - 2k, k)$ is any point on the line.

Hence, (–1, 2, 0), (0, 0, 1) and (–1/2, 1, 1/2) are the points on the line.

7. (1), (2)

$$3x - 6y + 2z + 5 = 0 \qquad \text{(i)}$$

$$-4x + 12y - 3z + 3 = 0 \qquad \text{(ii)}$$

Bisectors are $\frac{3x - 6y + 2z + 5}{\sqrt{9 + 36 + 4}} = \pm \frac{-4x + 12y - 3z + 3}{\sqrt{16 + 144 + 9}}$

The plane which bisects the angle between the planes that contains the origin.

$$13(3x - 6y + 2z + 5) = 7(-4x + 12y - 3z + 3)$$

$$67x - 162y + 47z + 44 = 0 \qquad \text{(iii)}$$

Further, $3 \times (-4) + (-6)(12) + 2 \times (-3) < 0$

Hence, the origin lies in the acute angle.

8. **(1), (4)**

The given lines intersect if

$$\begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & \\ & 2 & 1 \end{vmatrix} = 0 \Rightarrow \lambda = 0, -1.$$

9. **(1), (3)**

The required plane is parallel to the bisector of the given planes.

Bisectors are $\frac{x-y+z-3}{\sqrt{3}} = \pm\frac{x+y+z+4}{\sqrt{3}}$

or $2y+7=0$ and $2x+2y+1=0$. Hence, the planes are $y=0$ and $x+y=0$.

10. **(1), (4)**

The equation of a plane passing through the line of intersection of the x–y and y–z planes is

$$z+\lambda x=0, \quad \lambda \in R$$

This plane makes an angle 45° with the x–y plane ($z=0$). Thus,

$$\cos 45° = \frac{1}{\sqrt{1}\sqrt{\lambda^2+1}}$$

or $\lambda = \pm 1$

11. **(1), (2)**

The plane is equally inclined to the lines. Hence, it is perpendicular to the angle bisector of the vectors $2\hat{i}-2\hat{j}-\hat{k}$ and $8\hat{i}+\hat{j}-4\hat{k}$.

Vector along the angle bisectors of the vectors are

$$\frac{2\hat{i}-2\hat{j}-\hat{k}}{3} \pm \frac{8\hat{i}+\hat{j}-4\hat{k}}{9}$$

or $\frac{14\hat{i}-5\hat{j}-7\hat{k}}{9}$ and $\frac{-2\hat{i}-7\hat{j}+\hat{k}}{9}$.

Hence, the equations of the planes are

$$14x-5y-7z=0 \text{ or } 2x+7y-z=0$$

12. **(1), (3)**

For line $\frac{x-1}{1}=\frac{y}{-1}=\frac{z-5}{-1}$, point (1, 0, 5) lies on the plane. Also, the vector along the line $\hat{i}-\hat{j}-\hat{k}$ is perpendicular to the normal $\hat{i}+2\hat{j}-\hat{k}$ to the plane.

For line $\vec{r}=2\hat{i}-\hat{j}+4\hat{k}+\lambda(3\hat{i}+\hat{j}+5\hat{k})$, point (2, –1, 4) lies on the plane and vector $3\hat{i}+\hat{j}+5\hat{k}$ is perpendicular to the normal $\hat{i}+2\hat{j}-\hat{k}$.

Line $x-y+z=2x+y-z=0$ passes through the origin, which is not on the given plane.

13. **(2), (3)**

Volume of tetrahedron $ABCD = \frac{1}{6}|[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]|$

$= 1$ cubic units.

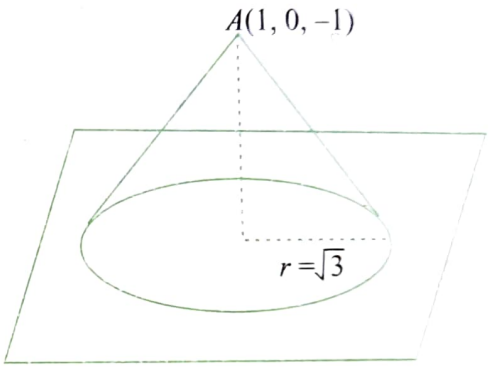
$$\Rightarrow \quad \begin{vmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ x-0 & y-1 & z-2 \end{vmatrix} = \pm 6$$

or $-2(y-1)-2(z-2)=\pm 6$

or $y-1+z-2=\pm 3$

$\Rightarrow$ $y+z=6$ or $y+z=0$

14. **(1), (3), (4)**



The rod sweeps out the figure which is a cone.

The distance of point $A(1, 0, -1)$ from the plane is

$$\frac{|1-2+4|}{\sqrt{9}} = 1 \text{ unit}$$

The slant height l of the cone is 2 units.

Then the radius of the base of the cone is

$$\sqrt{l^2-1} = \sqrt{4-1} = \sqrt{3}$$

Hence, the volume of the cone is

$$\frac{\pi}{3}(\sqrt{3})^2 \cdot 1 = \pi \text{ cubic units}$$

Area of the circle on the plane which the rod traces is 3π.

Also, the centre of the circle is $Q(x, y, z)$. Then

$$\frac{x-1}{1}=\frac{y-0}{-2}=\frac{z+1}{2}=\frac{-(1-0-2+4)}{1^2+(-2)^2+2^2}$$

or $Q(x, y, z) \equiv \left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right)$

15. **(2), (3)**

Distance between the planes is $h = 5\sqrt{6}$.

Also the figure formed is a cylinder, whose radius is $r = 2$ units.

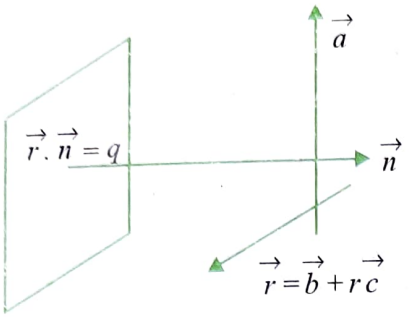
Hence, the volume of the cylinder is

$$\pi r^2 h = \pi(2)^2 \cdot \frac{5}{\sqrt{6}} = \frac{20\pi}{\sqrt{6}} \text{ cubic units}$$

Also the curved surface area is

$$2\pi rh = 2\pi(2)\cdot\frac{5}{\sqrt{6}} = \frac{20\pi}{\sqrt{6}}$$

16. (1), (2)



Required line is parallel to $\vec{n} \times \vec{c}$

The equation of line is $\vec{r} = \vec{a} + \lambda(\vec{n} \times \vec{c})$

or $(\vec{r} - \vec{a}) = \lambda(\vec{n} \times \vec{c})$

$\therefore$ $(\vec{r} - \vec{a}) \times (\vec{n} \times \vec{c}) = 0$

17. (1), (2), (3)

$$x + y + z - 1 = 0$$
$$4x + y - 2z + 2 = 0$$

Therefore, direction ratios of the line are (–3, 6, –3), i.e., (1, –2, 1)

Let $z = k$, then $x = k - 1$, $y = 2 - 2k$

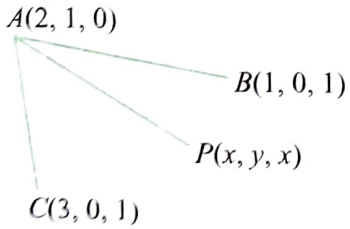
i.e., $(k - 1, 2 - 2k, k)$ is any point on the line

Therefore, (–1, 2, 0), (0, 0, 1) and $\left(-\frac{1}{2}, 1, \frac{1}{2}\right)$ are points on the lines.

Linked Comprehension Type

For Problems 1–3

1. (2), 2. (3), 3. (4)



$$\begin{vmatrix} x-2 & y-1 & z \\ 1-2 & 0-1 & 1-0 \\ 3-2 & 0-1 & 1-0 \end{vmatrix} = 0$$

$$(x-2)[(-1)-(-1)] - (y-1)[(-1)-1] + z[1+1] = 0$$

or $2(y-1) + 2z = 0$

or $y + z - 1 = 0$

The vector normal to the plane is $\vec{r} = 0\hat{i} + \hat{j} + \hat{k}$.

The equation of the line through (0, 0, 2) and parallel to $\vec{n}$ is $\vec{r} = 2\hat{k} + \lambda(\hat{j} + \hat{k})$.

The perpendicular distance of $D(0, 0, 2)$ from plane ABC is $\left|\frac{2-1}{\sqrt{1^2+1^2}}\right| = \frac{1}{\sqrt{2}}$.

For Problems 4–6

4. (2) Let $Q(x_2, y_2, z_2)$ be the image of $A(2, 1, 6)$ about mirror $x + y - 2z = 3$. Then,

$$\frac{x_2 - 2}{1} = \frac{y_2 - 1}{1} = \frac{z_2 - 6}{-2}$$
$$= \frac{-2(2 + 1 - 12 - 3)}{1^2 + 1^2 + 2^2} = 4$$

$\Rightarrow$ $(x_2, y_2, z_2) \equiv (6, 5, -2)$.

5. (3) Let $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-6}{5} = \lambda$

$x = 2 + 3\lambda, y = 1 + 4\lambda, z = 6 + 5\lambda$ lies on plane $x + y - 2z = 3$

$\Rightarrow$ $2 + 3\lambda + 1 + 4\lambda - 2(6 + 5\lambda) = 3$

or $3 + 7\lambda - 12 - 10\lambda = 3$ or $-3\lambda = 12$ or $\lambda = -4$

Point $B \equiv (-10, -15, -14)$

6. (3) The equation of the reflected ray $L_1 = 0$ is the line joining $Q(x_2, y_2, z_2)$ and $B(-10, -15, -14)$.

$$\frac{x+10}{16} = \frac{y+15}{20} = \frac{z+14}{12}$$

or $\frac{x+10}{4} = \frac{y+15}{5} = \frac{z+14}{3}$

For Problems 7–9

7. (2), 8. (3), 9. (2)

The given system of equations is

$$2x + py + 6z = 8$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

$$\Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = (2 - p)(3 - q)$$

By Cramer's rule, if $\Delta \neq 0$, i.e., $p \neq 2$ and $q \neq 3$, the system has a unique solution.

If $p = 2$ or $q = 3$, $\Delta = 0$, then if $\Delta_x = \Delta_y = \Delta_z = 0$, the system has infinite solutions and if any one of Δ_x, Δ_y and $\Delta_z \neq 0$, the system has no solution.

Now $\Delta_x = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix}$

$= 30 - 8q - 15p + 4pq = (4q - 15) \cdot (p - 2)$

$\Delta_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix}$

$= -8q + 8q = 0$

$\Delta_z = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix}$

$= p - 2$

Thus, if $p = 2$, $\Delta_x = \Delta_y = \Delta_z = 0$ for all $q \in R$, the system has infinite solutions.

If $p \neq 2$, $q = 3$ and $\Delta_z \neq 0$, then the system has no solution.

Hence the system has (i) no solution if $p \neq 2$ and $q = 3$, (ii) a unique solution if $p \neq 2$ and $q \neq 3$ and (iii) infinite solutions if $p = 2$ and $q \in R$.

For Problems 10–12

10. (4) The line $\frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{4} = r$

Any point say $B \equiv (3r + 1, 2 - r, 3 + 4r)$ (on the line L)

$\overrightarrow{AB} = 3r, -r, 4r + 6$

Hence,

$\overrightarrow{AB}$ is parallel to $x + y - z = 1$

$\Rightarrow$ $3r - r - 4r - 6 = 0$ or $r = -3$

B is $(-8, 5, -9)$

11. (2) The equation of plane containing the line L is

$A(x - 1) + B(y - 2) + C(z - 3) = 0$,

where $3A - B + 4C = 0$ (i)

(i) also contains point $A(1, 2, -3)$.

Hence, $C = 0$ and $3A = B$.

The equation of plane is $x - 1 + 3(y - 2) = 0$ or $x + 3y - 7 = 0$

12. (4) The distance of point $(1 + 3r, 2 - r, 3 + 4r)$ from the planeis $\frac{|1 + 3r + 2 - r - 3 - 4r - 1|}{\sqrt{1+1+1}} = \frac{|2r + 1|}{\sqrt{3}} = \frac{4}{\sqrt{3}}$

$\Rightarrow$ $r = \frac{3}{2}, -\frac{5}{2}$

Hence, the points are $A\left(\frac{11}{2}, \frac{1}{2}, \frac{10}{2}\right)$ and $B\left(\frac{-13}{2}, \frac{9}{2}, \frac{-14}{2}\right)$.

$\Rightarrow$ $AB = \sqrt{292}$

Matrix Match Type

1. a → s; b→ q; c → p; d → r

a. Line $x = 2t + 1$, $y = t + 2$, $z = -t - 3$ or $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{-1}$, which is along the vector $2\hat{i} + \hat{j} - \hat{k}$. Vector $\hat{i} + 3\hat{j} + 5\hat{k}$ is perpendicular to the line.

b. Normals to the planes $x + y + z - 3 = 0$ and $2x - y + 3z = 0$ are $\vec{n_1} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{n_2} = 2\hat{i} - \hat{j} + 3\hat{k}$

Then the vector along the line of intersection of planes is $\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 4\hat{i} - \hat{j} - 3\hat{k}$

c. The shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{-1}$ and $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} - 2\hat{k})$ occurs along the vector

$(2\hat{i} - 3\hat{j} - \hat{k}) \times (\hat{i} + \hat{j} - 2\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -1 \\ 1 & 1 & -2 \end{vmatrix}$

$= 7\hat{i} + 3\hat{j} + 5\hat{k}$

d. Normal to the plane $\vec{r} = -\hat{i} + 4\hat{j} - 6\hat{k} + \lambda(\hat{i} + 3\hat{j} - 2\hat{k}) + \mu(-\hat{i} + 2\hat{j} - 5\hat{k})$ is

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 2 & -5 \end{vmatrix} = -11\hat{i} + 7\hat{j} + 5\hat{k}$

2. a → q, s; b → r; c → p, q; d → p

a. Line $\frac{x-1}{-2} = \frac{y+2}{3} = \frac{z}{-1}$ is along the vector $\vec{a} = -2\hat{i} + 3\hat{j} - \hat{k}$ and line $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} + \hat{k})$ is along the vector $\vec{b} = \hat{i} + \hat{j} + \hat{k}$. Here $\vec{a} \perp \vec{b}$.

Also $\begin{vmatrix} 3-1 & -1-(-2) & 1-0 \\ -2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$

b. The direction ratios of the line $x - y + 2z - 4 = 0 = 2x + y - 3z + 5 = 0$ are $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i} + 7\hat{j} + 3\hat{k}$.

Hence, the given two lines are parallel.

c. The given lines are $(x = t - 3, y = -2t + 1, z = -3t - 2)$ and $\vec{r} = (t+1)\hat{i} + (2t+3)\hat{j} + (-t-9)\hat{k}$, or

$$\frac{x+3}{1} = \frac{y-1}{-2} = \frac{z+2}{-3} \text{ and } \frac{x-1}{1} = \frac{y-3}{2} = \frac{z+9}{-1}.$$

The lines are perpendicular as

$$(1)(1) + (-2)(2) + (-3)(-1) = 0.$$

Also $\begin{vmatrix} -3-1 & 1-3 & -2-(-9) \\ 1 & -2 & -3 \\ 1 & 2 & -1 \end{vmatrix} = 0$

Hence, the lines are intersecting.

d. The given lines are $\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} - \hat{k})$ and $\vec{r} = (-\hat{i} - 2\hat{j} + 5\hat{k}) + s(\hat{i} - 2\hat{j} + \frac{3}{4}\hat{k})$.

$$\begin{vmatrix} 1-(-1) & 3-(-2) & -1-5 \\ 2 & -1 & -1 \\ 1 & -2 & 3/4 \end{vmatrix} = 0$$

Hence, the lines are coplanar and intersecting (as the lines are not parallel).

3. a → q; b → p; c → s; d → r

a. The given line is $x = 4y + 5, z = 3y - 6$

or $\frac{x-5}{4} = y, \frac{z+6}{3} = y$

or $\frac{x-5}{4} = \frac{y}{1} = \frac{z+6}{3} = \lambda$ (say)

Any point on the line is of the form $(4\lambda + 5, \lambda, 3\lambda - 6)$.

The distance between $(4\lambda + 5, \lambda, 3\lambda - 6)$ and $(5, 3, -6)$ is 3 units (given). Therefore

$$(4\lambda + 5 - 5)^2 + (\lambda - 3)^2 + (3\lambda - 6 + 6)^2 = 9$$

or $16\lambda^2 + \lambda^2 + 9 - 6\lambda + 9\lambda^2 = 9$

or $26\lambda^2 - 6\lambda = 0$

or $\lambda = 0, 3/13$

The point is $(5, 0, -6)$.

b. The equation of the plane containing the lines $\frac{x-2}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and parallel to $\hat{i} + 4\hat{j} + 7\hat{k}$ is

$$\begin{vmatrix} x-2 & y+3 & z+5 \\ 1 & 4 & 7 \\ 3 & 5 & 7 \end{vmatrix} = 0$$

or $x - 2y + z - 3 = 0$

Point $(-1, -2, 0)$ lies on this plane.

c. The line passing through points $A(2, -3, -1)$ and $B(8, -1, 2)$ is $\frac{x-2}{8-2} = \frac{y+3}{-1+3} = \frac{z+1}{2+1}$

or $\frac{x-2}{6} = \frac{y+3}{2} = \frac{z+1}{3} = \lambda$ (say).

Any point on this line is of the form $P(6\lambda + 2, 2\lambda - 3, 3\lambda - 1)$, whose distance from point $A(2, -3, -1)$ is 14 units. Therefore,

$PA = 14$

or $PA^2 = (14)^2$

$\Rightarrow$ $(6\lambda)^2 + (2\lambda)^2 + (3\lambda)^2 = 196$

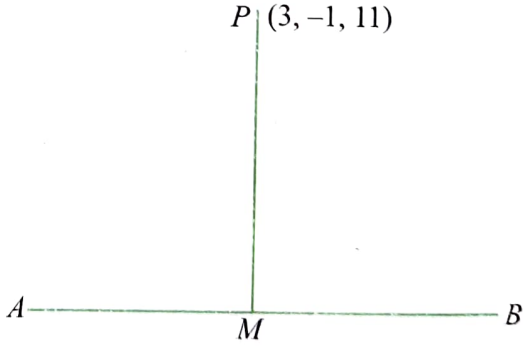
or $49\lambda^2 = 196$

or $\lambda^2 = 4$

or $\lambda = \pm 2$

Therefore, the required points are (14, 1, 5) and $(-10, -7, -7)$. The point nearer to the origin is (14, 1, 5).

d. Any point on line AB, $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$ is $M(2\lambda, 3\lambda + 2, 4\lambda + 3)$. Therefore, the direction ratios of PM are $2\lambda - 3$, $3\lambda + 3$ and $4\lambda - 8$.



But $PM \perp AB$

$\therefore$ $2(2\lambda - 3) + 3(3\lambda + 3) + 4(4\lambda - 8) = 0$

$4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$

$29\lambda - 29 = 0$

$\lambda = 1$

Therefore, foot of the perpendicular is $M(2, 5, 7)$.

$\rightarrow$ r; b $\rightarrow$ p; c $\rightarrow$ q; d $\rightarrow$ s

a. The given line and plane are $\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$, respectively. Since $(\hat{i} - \hat{j} + 4\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 0$, line and plane are parallel.

Hence, the required distance is equal to distance of point $(2, -2, 3)$ from the plane $x + 5y + z - 5 = 0$, which is $\dfrac{|2 - 10 + 3 - 5|}{\sqrt{1 + 25 + 1}} = \dfrac{10}{3\sqrt{3}}$

b. The distance between two parallel planes $\vec{r} \cdot (2i - j + 3k) = 4$ and $\vec{r} \cdot (6i - 3j + 9k) + 13 = 0$ is

$$d = \frac{|4 - (-13/3)|}{\sqrt{(2)^2 + (-1)^2 + (3)^2}} = \frac{(25/3)}{\sqrt{14}}$$

$$= \frac{25}{3\sqrt{14}}$$

c. The perpendicular distance of the point $(2, 5, -3)$ from the plane $\vec{r} \cdot (6i - 3j + 2k) = 4$ or $6x - 3y + 2z - 4 = 0$ is

$$d = \frac{|12 - 15 - 6 - 4|}{\sqrt{36 + 9 + 4}}$$

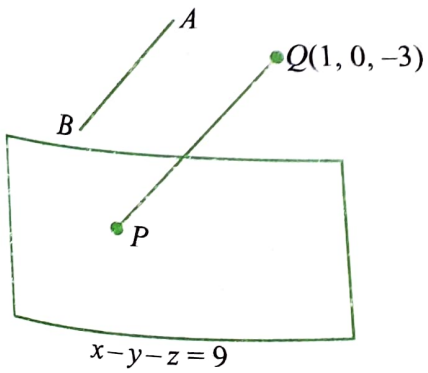
$$= 13/\sqrt{49} = 13/7$$

d. The equation of the line AB is

$$\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$$

The equation of line passing through $(1, 0, -3)$ and parallel to AB is

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} = r \quad \text{(say)}$$



The coordinates of any point on line $P(2r + 1, 3r, -6r - 3)$ which lie on plane

$$(2r + 1) - (3r) - (-6r - 3) = 9$$

$$r = 1$$

Point $P \equiv (3, 3, -9)$

Required distance

$$PQ = \sqrt{(3-1)^2 + (3-0)^2 + (-9+3)^2}$$

$$= \sqrt{4 + 9 + 36} = 7$$

5. a $\rightarrow$ q; b $\rightarrow$ r; c $\rightarrow$ s; d $\rightarrow$ p

a. If the required image is (x, y, z), then

$$\frac{x-3}{2} = \frac{y-5}{1} = \frac{z-7}{1} = -\frac{2(6+5+7+18)}{2^2 + 1^2 + 1^2} = -12$$

or $(x, y, z) \equiv (-21, -7, -5)$.

b. Any point on the line $\dfrac{x-2}{-3} = \dfrac{y-1}{2} = \dfrac{z-3}{2} = \lambda$ is $(-3\lambda + 2, 2\lambda + 1, 2\lambda + 3)$, which lies on plane $2x + y - z = 3$. Therefore

$$-6\lambda + 4 + 2\lambda + 1 - 2\lambda - 3 = 3$$

$$-6\lambda = 1$$

$$\lambda = -1/6$$

Therefore, the point is $\left(\dfrac{5}{2}, \dfrac{2}{3}, \dfrac{8}{3}\right)$.

c. If (x, y, z) is the required foot of the perpendicular, then $\dfrac{x-1}{2} = \dfrac{y-1}{-2} = \dfrac{z-2}{4} = -\dfrac{(2-2+8+5)}{2^2 + (-2)^2 + 4^2}$

or $\quad (x, y, z) \equiv \left(\dfrac{-1}{12}, \dfrac{25}{12}, \dfrac{-2}{12}\right)$

d. Any point on the line $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4} = \lambda$ is $P(2\lambda + 1, 3\lambda + 2, 4\lambda + 3)$, which satisfies the line $\dfrac{x-4}{5} = \dfrac{y-1}{2} = \dfrac{z}{1}$

or $\quad \dfrac{2\lambda + 1 - 4}{5} = \dfrac{3\lambda + 2 - 1}{2} = \dfrac{4\lambda + 3}{1}$

or $\quad \lambda = -1$

The required point is $(-1, -1, -1)$

Numerical Value Type

1. (1) If image of point $(2, -3, 3)$ in the plane $x - 2y - z + 1 = 0$ is (a, b, c), then

$$\frac{a-2}{1} = \frac{b+3}{-2} = \frac{c-3}{-1} = \frac{-2(2 - 2(-3) - 3 + 1)}{(1)^2 + (-2)^2 + (-1)^2} = -2$$

Hence, the image is (0, 1, 5)

Obviously distance of image of the point from the z-axis is 1.

2. (3) Let $A(1, 0, -1)$, $B(-1, 2, 2)$

Direction ratios of AB are $(2, -2, -3)$

Let θ be the angle between the line and normal to plane; then

$$\cos\theta = \frac{|2.1 + 3(-2) - 5(-3)|}{\sqrt{1+9+25}\sqrt{4+4+9}}$$

$$= \frac{11}{\sqrt{17}\sqrt{35}} = \frac{11}{\sqrt{595}}$$

Length of projection

$$= (AB)\sin\theta$$

$$= \sqrt{(2)^2 + (2)^2 + (3)^2} \times \sqrt{1 - \frac{121}{595}}$$

$$= \sqrt{\frac{474}{35}} \text{ units}$$

3. (2) Vector normal to the plane is $\vec{n} = \hat{i} - 3\hat{j} + 2\hat{k}$ and vector along the line is $\vec{v} = 2\hat{i} + \hat{j} - 3\hat{k}$

Now $\sin\theta = \dfrac{\vec{x}\cdot\vec{v}}{|\vec{x}||\vec{v}|} = \left|\dfrac{2-3-6}{\sqrt{14}\sqrt{14}}\right| = \left|\dfrac{7}{14}\right|$

Hence, $\operatorname{cosec}\theta = 2$

4. (6) A plane containing the line of intersection of the given planes is

$$x - y - z - 4 + \lambda(x + y + 2z - 4) = 0$$

i.e., $(\lambda + 1)x + (\lambda - 1)y + (2\lambda - 1)z - 4(\lambda + 1) = 0$

vector normal to it

$$V = (\lambda + 1)\hat{i} + (\lambda - 1)\hat{j} + (2\lambda - 1)\hat{k}S \qquad \text{(i)}$$

Now the vector along the line of intersection of the planes $2x + 3y + z - 1 = 0$ and $x + 3y + 2z - 2 = 0$ is given by

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3(\hat{i} - \hat{j} + \hat{k})$$

As $\vec{n}$ is parallel to the plane (i), we have

$$\vec{n}\cdot\vec{V} = 0$$

$$(\lambda + 1) - (\lambda - 1) + (2\lambda - 1) = 0$$

$$2 + 2\lambda - 1 = 0 \quad \text{or} \quad \lambda = \frac{-1}{2}$$

Hence, the required plane is

$$\frac{x}{2} - \frac{3y}{2} - 2z - 2 = 0$$

$$x - 3y - 4z - 4 = 0$$

Hence, $|A + B + C| = 6$

5. (7) Clearly minimum value of $a^2 + b^2 + c^2$

$$= \left(\frac{|(3(0) + 2(0) + (0) - 7|}{\sqrt{(3)^2 + (2)^2 + (1)^2}}\right)^2 = \frac{49}{14} = \frac{7}{2} \text{ units}$$

6. (7)

$$4x + 7y + 4z + 81 = 0 \qquad \text{(i)}$$

$$5x + 3y + 10z = 25 \qquad \text{(ii)}$$

Equation of plane passing through their line of intersection is

$$(4x + 7y + 4z + 81) + \lambda(5x + 3y + 10z - 25) = 0$$

or $(4 + 5\lambda)x + (7 + 3\lambda)y + (4 + 10\lambda)z + 81 - 25\lambda = 0 \qquad \text{(iii)}$

Plane (iii) $\perp$ to (i), so

$$4(4 + 5\lambda) + 7(7 + 3\lambda) + 4(4 + 10\lambda) = 0$$

$\therefore \quad \lambda = -1$

From (iii), equation of plane is

$$-x + 4y - 6z + 106 = 0 \qquad \text{(iv)}$$

Distance of (iv) from $(0,0,0) = \dfrac{106}{\sqrt{1+16+36}} = \dfrac{106}{\sqrt{53}}$

7. (9) Line through point $P(-2, 3, -4)$ and parallel to the given line $\dfrac{x+2}{3} = \dfrac{2y+3}{4} = \dfrac{3z+4}{5}$ is

$$\frac{x+2}{3} = \frac{y+\frac{3}{2}}{2} = \frac{z+\frac{4}{3}}{\frac{5}{3}} = \lambda$$

Any point on this line is

$$Q\left[3\lambda - 2, 2\lambda - \frac{3}{2}, \frac{5}{3}\lambda - \frac{4}{3}\right]$$

Direction ratios of PQ are $\left[3\lambda, \dfrac{4\lambda - 9}{2}, \dfrac{5\lambda + 8}{3}\right]$

Now PQ is parallel to the given plane

$$4x + 12y - 3z + 1 = 0$$

Hence, line is perpendicular to the normal to the plane. Thus,

$$4(3\lambda) + 12\left(\frac{4\lambda - 9}{2}\right) - 3\left(\frac{5\lambda + 8}{3}\right) = 0$$

or $\lambda = 2$

$$\Rightarrow \quad Q \equiv \left(4, \frac{5}{2}, 2\right)$$

$$\Rightarrow \quad PQ = \sqrt{(6)^2 + \left(\frac{5}{2} - 3\right)^2 + (6)^2} = \frac{17}{2}$$

8. (6) The given points are $O(0, 0, 0)$, $A(0, 0, 2)$, $B(0, 4, 0)$ and $C(6, 0, 0)$

Here three faces of tetrahedron are xy, yz, zx plane.

Since point P is equidistance from zx, xy and yz planes, its coordinates are $P(r, r, r)$

Equation of plane ABC is

$$2x + 3y + 6z = 12 \text{ (from intercept form)}$$

P is also at distance r from plane ABC. Thus,

$$\frac{|2r + 3r + 6r - 12|}{\sqrt{4+9+36}} = r$$

or $\quad |11r - 12| = 7r$

or $\quad 11r - 12 = \pm 7r$

or $\quad r = \frac{12}{18}, 3$

$\therefore \quad r = 2/3$ (as $r < 2$)

9. (7) As the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$, the point $(4, 2, k)$ through which it passes must also lie on the given plane, and hence $2 \times 4 - 4 \times 2 + k = 7$ or $k = 7$.

10. (4.5) $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = \lambda$

$\Rightarrow \quad x = 2\lambda + 1, y = 3\lambda - 1$ and $z = 4\lambda + 1$

$$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = \mu$$

$\Rightarrow \quad x = 3 + \mu, y = k + 2\mu$ and $z = \mu$

Since the above lines intersect,

$$2\lambda + 1 = 3 + \mu \quad \text{(i)}$$

$$3\lambda - 1 = 2\mu + k \quad \text{(ii)}$$

$$\mu = 4\lambda + 1 \quad \text{(iii)}$$

Solving (i) and (iii) and putting the value of λ and μ in (ii), $k = 9/2$.

11. (9) Let the equation of the variable plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, which meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

The centroid of ΔABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ and it satisfies the relation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$. Thus,

$$\frac{9}{a^2} + \frac{9}{b^2} + \frac{9}{c^2} = k$$

or $\quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{k}{9} \quad \text{(i)}$

Also it is given that the distance of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ from $(0, 0, 0)$ is 1 unit. Therefore,

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = 1 \text{ or } \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 1 \quad \text{(ii)}$$

From (i) and (ii), we get $k/9 = 1$, i.e., $k = 9$

12. (8) The equation of the plane passing through the point $(1, -2, 1)$ and perpendicular to the planes $2x - 2y + z = 0$ and $x - y + 2z = 4$ is given by

$$\begin{vmatrix} x-1 & y+2 & z-1 \\ 2 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 0$$

or $\quad x + y + 1 = 0$

Its distance from the point $(1, 2, 2)$ is

$$\left|\frac{1+2+1}{\sqrt{2}}\right| = 2\sqrt{2}$$

Archives

JEE Main

Single Correct Answer Type

1. (2) The d.r.'s of the line are $(3, -5, 2)$.

The d.r.'s of the normal to the plane are $(1, 3, -\alpha)$.

Since line lies on the plane, the line is perpendicular to the normal of the plane. So

$$3(1) - 5(3) + 2(-\alpha) = 0$$

$\Rightarrow \quad 3 - 15 - 2\alpha = 0$

$\Rightarrow \quad 2\alpha = -12 \Rightarrow \alpha = -6$

Also $(2, 1, -2)$ lies on the plane. So

$$2 + 3 + 6(-2) + \beta = 0 \Rightarrow \beta = 7$$

$\therefore \quad (\alpha, \beta) = (-6, 7)$

2. (3) The projection of a vector coordinate axis are

$$x_2 - x_1, y_2 - y_1, z_2 - z_1$$

$$x_2 - x_1 = 6, y_2 - y_1 = -3, z_2 - z_1 = 2$$

$\therefore \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$= \sqrt{36 + 9 + 4} = 7$$

The direction cosines of the vector are $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

3. (3) $A(3, 1, 6), B = (1, 3, 4)$

The midpoint of AB, $(2, 2, 5)$, lies on the plane.

The d.r.'s of AB are $(2, -2, 2)$.

The d.r.'s of the normal to the plane are $(1, -1, 1)$.

AB is the perpendicular bisector. So A is the image of B.

Statement 2 is correct but it is not a correct explanation of statement 1.

4. (4) $l = \cos 45° = \frac{1}{\sqrt{2}}$

$m = \cos 120° = -\frac{1}{2}$

$n = \cos\theta,$

where θ is the angle which the line makes with the positive z-axis.

Now $l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$

$\Rightarrow \quad \cos^2\theta = \frac{1}{4}$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$ (θ being acute)

$\Rightarrow \quad \theta = \frac{\pi}{3}$

5. (3) The midpoint of $A(1, 0, 7)$ and $B(1, 6, 3)$ is $(1, 3, 5)$ which lies on the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Direction ratios of AB are $(1,6,-4)$.

Also the line passing through the points A and B is perpendicular to the given line, hence B is the mirror image of A, consequently statement 1 is true.

Statement 2 is also true but it is not a correct explanation of statement 1 as there are infinitely many lines passing through the midpoint of the line segment and one of the lines is perpendicular bisector.

6. (2) The direction ratios of the given line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-3}{\lambda}$ are $1, 2, \lambda$.

$\Rightarrow$ Direction cosines of the line are

$$\frac{1}{\sqrt{5+\lambda^2}}, \frac{2}{\sqrt{5+\lambda^2}}, \frac{\lambda}{\sqrt{5+\lambda^2}}$$

Also the direction cosines of the normal to the plane are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

Angle between the line and the plane is θ.

Then $\cos(90° - \theta) = \frac{1+4+3\lambda}{\sqrt{(5+\lambda^2)}\sqrt{14}} = \sin\theta$

$\Rightarrow \frac{3}{\sqrt{14}} = \frac{5+3\lambda}{\sqrt{\lambda^2+5}\sqrt{14}}$ $\left(\because \text{given } \theta = \cos^{-1}\sqrt{\frac{5}{14}}\right)$

$\Rightarrow \quad 9\lambda^2 + 45 = 9\lambda^2 + 30\lambda + 25$

$\Rightarrow \quad 30\lambda = 20$

$\Rightarrow \quad \lambda = \frac{2}{3}$

7. (1) Equation of plane parallel to the given plane is $x - 2y + 2z + d = 0$

Its distance from origin is 1.

So, $\left|\frac{d}{\sqrt{1^2+(-2)^2+2^2}}\right| = 1$

$d = \pm 3$

So equation required plane is $x - 2y + 2z \pm 3 = 0$.

8. (3) Given lines are $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$

$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$

These lines will intersect if lines are coplanar.

$\therefore \quad \begin{vmatrix} 3-1 & k+1 & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad 2(-5) - (k+1)(-2) - 1(1) = 0$

$\Rightarrow \quad 2(k+1) = 11$

$\Rightarrow \quad k = \frac{9}{2}$

9. (3) Planes are $2x + y + 2z - 8 = 0$ and

$2x + y + 2z + \frac{5}{2} = 0$

$$\text{Distance} = \frac{\frac{5}{2} - (-8)}{\sqrt{2^2+1^2+2^2}} = \frac{\frac{21}{2}}{3} = \frac{7}{2}$$

10. (3) Lines are coplanar

$\Rightarrow \quad \begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad (1 + 2k) + (1 + k^2) - (2 - k) = 0$

$\Rightarrow \quad 3k + k^2 = 0$

$\Rightarrow \quad k(k+3)=0$

$\Rightarrow \quad k=0 \text{ or } k=-3$

Exactly two values.

11. (1) Line is parallel to plane.

Now, image of (1, 3, 4) in plane is (–3, 5, 2).

So, equation of line is $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$.

12. (3) Equation of planes through given line is

$$2x-5y+z-3+\lambda(x+y+4z-5)=0$$

$\therefore \quad x(2+\lambda)+y(\lambda-5)+z(4\lambda+1)-3-5\lambda=0$

This plane is parallel to $x+3y+6z=1$

$\Rightarrow \quad \frac{\lambda+2}{1}=\frac{\lambda-5}{3}=\frac{4\lambda+1}{6}$

$\Rightarrow \quad 3\lambda+6=\lambda-5$

$\Rightarrow \quad 2\lambda=-11$

$\Rightarrow \quad \lambda=\frac{-11}{2}$

Therefore, equation of plane is

$$\frac{-7x}{2}-\frac{21y}{2}-21z+\frac{49}{2}=0$$

$\Rightarrow \quad 7x+21y+42z-49=0$

$\Rightarrow \quad x+3y+6z-7=0$

13. (3) Given line lies in the plane $lx+my-z=9$

$\therefore$ Point $(3,-2,-4)$ lies on the plane

$\therefore \quad 3l-2m+4=0$

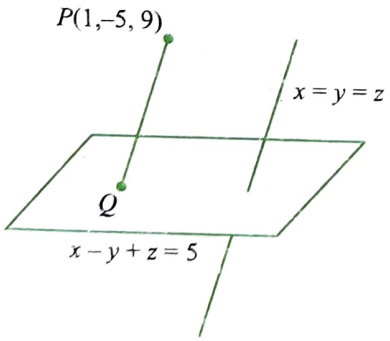
$\Rightarrow \quad 3l-2m=5 \qquad$ (i)

Also $2l-m-3=0$

$\Rightarrow \quad 2l-m=3 \qquad$ (ii)

From (i) and (ii), $l=1$ and $m=-1$

14. (1)



Equation of line PQ : $\frac{x-1}{1}=\frac{y+5}{1}=\frac{z-9}{1}=\lambda$

$\therefore \quad Q$ can be taken as $(\lambda+1, \lambda-5, \lambda+9)$

As Q lies on plane $x-y+z=5$

$\therefore \quad (\lambda+1)-(\lambda-5)+(\lambda+9)=5$

$\therefore \quad \lambda=-10$

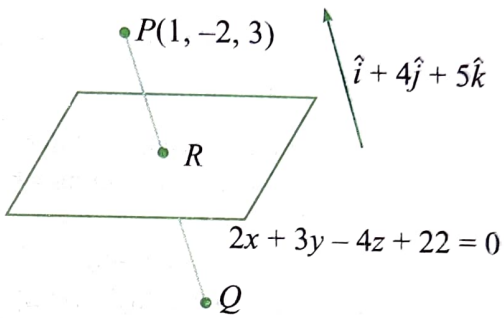
$\therefore$ Point Q is $(-9,-15,-1)$

$\therefore$ Required distance

$$PQ=\sqrt{(1+9)^2+(-5+15)^2+(9+1)^2}$$

$$=\sqrt{100+100+100}=10\sqrt{3}$$

15. (3)



Equation of line PQ is $\frac{x-1}{1}=\frac{y+2}{4}=\frac{z-3}{5}$.

Let $\quad R(\lambda+1, 4\lambda-2, 5\lambda+3)$.

Since R lies on the plane $2x+3y-4z+22=0$, we have

$$2(\lambda+1)+3(4\lambda-2)-4(5\lambda+3)+22=0$$

$\Rightarrow \quad -6\lambda+6=0 \quad \Rightarrow \lambda=1$

$\therefore \quad R \equiv (2, 2, 8)$

$\therefore \quad PQ = 2PR = 2\sqrt{42}$

16. (3) The plane is parallel to the given two lines.

So, vector normal to the plane is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix} = 5\hat{i}+7\hat{j}+3\hat{k}$$

The plane passes through the point $(1,-1,-1)$.

So, equation of the plane is

$$5(x-1)+7(y+1)+3(z+1)=0$$

or $\quad 5x+7y+3z+5=0$

Distance of this plane from $(1, 3, -7)$

$$=\frac{|5+21-21+5|}{\sqrt{25+49+9}}=\frac{10}{\sqrt{83}}$$

17. (3) Given planes are:

$$2x-2y+3z-2=0 \qquad \text{(i)}$$

$$x-y+z+1=0 \qquad \text{(ii)}$$

$$x+2y-z-3=0 \qquad \text{(iii)}$$

$$3x-y+2z-1=0 \qquad \text{(iv)}$$

Vector along line of intersection (L_1) of plane (i) and (ii) is

$$\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i} + \hat{j}$$

Vector along line of intersection (L_2) of planes (iii) and (iv) is

$$\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 3\hat{i} - 5\hat{j} - 7\hat{k}$$

For point on the plane containing lines L_1 and L_2, in first two equations of plane putting $x = 0$, we get $y = 5$ and $z = 4$.

So, point on the plane is (0, 5, 4).

$\therefore$ Equation of plane containing lines L_1 and L_2 is

$$\begin{vmatrix} x & y-5 & z-4 \\ 1 & 1 & 0 \\ 3 & -5 & -7 \end{vmatrix} = 0$$

or $7x - 7y + 8z + 3 = 0$

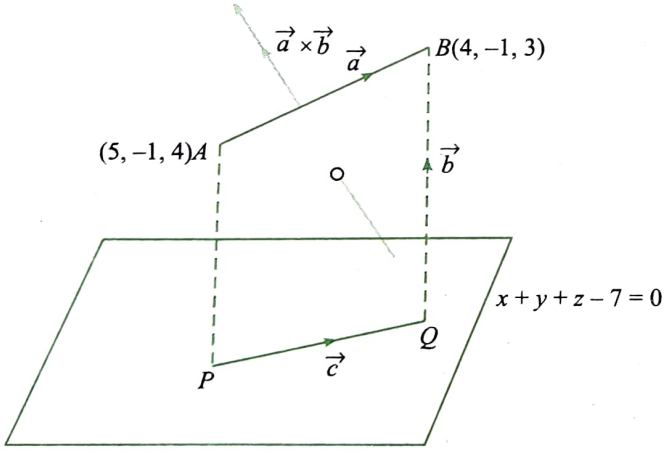
Distance of origin from this plane

$$= \frac{|0 - 0 + 0 + 3|}{\sqrt{162}} = \frac{3}{9\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

18. (1) In the figure PQ is projection of AB on the given plane

$$x + y + z - 7 = 0$$

PQ is line of intersection of given plane and plane $ABQP$.



Normal to given plane is $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.

Vector $\overrightarrow{AB} = \vec{a} = -\hat{i} - \hat{k}$

Normal to plane $ABQP$ is $\vec{a} \times \vec{b}$.

$\therefore$ Vector $\overrightarrow{PQ} = \vec{c} = (\vec{a} \times \vec{b}) \times \vec{b} = (\vec{a} \cdot \vec{b})\vec{b} - |\vec{b}|^2 \vec{a}$

$$= -2(\hat{i} + \hat{j} + \hat{k}) - 3(-\hat{i} - \hat{k}) = \hat{i} - 2\hat{j} + \hat{k}$$

$$\therefore PQ = \frac{|\vec{a} \cdot \vec{c}|}{|\vec{c}|} = \frac{|-1 - 1|}{\sqrt{6}} = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$

JEE Advanced

Single Correct Answer Type

1. (1) Any point on the line can be taken as

$$Q = \{(1 - 3\mu), (\mu - 1), (5\mu + 2)\}$$

$$\overrightarrow{PQ} = \{-3\mu - 2, \mu - 3, 5\mu - 4)$$

Now, $1(-3\mu - 2) - 4(\mu - 3) + 3(5\mu - 4) = 0$

or $-3\mu - 2 - 4\mu + 12 + 15\mu - 12 = 0$

or $8\mu = 2$ or $\mu = 1/4$

2. (3) Plane 1: $ax + by + cz = 0$ contains line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$

$\therefore$ $2a + 3b + 4c = 0$ (i)

Plane 2: $a'x + b'y + c'z = 0$ is perpendicular to plane containing lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$

$\therefore$ $3a' + 4b' + 2c' = 0$ and $4a' + 2b' + 3c' = 0$

$\Rightarrow$ $\frac{a'}{12 - 4} = \frac{b'}{8 - 9} = \frac{c'}{6 - 16}$

$\Rightarrow$ $8a - b - 10c = 0$ (ii)

From (i) and (ii),

$$\frac{a}{-30 + 4} = \frac{b}{32 + 20} = \frac{c}{-2 - 24}$$

$\Rightarrow$ Equation of plane $x - 2y + z = 0$

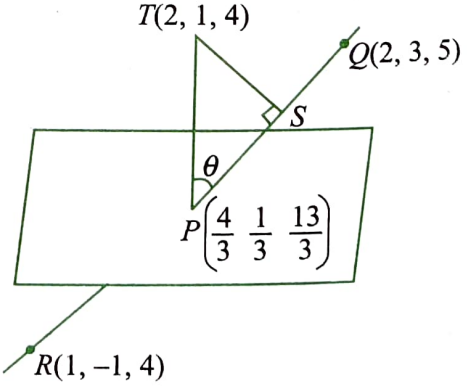
3. (1) Distance of point $(1, -2, 1)$ from plane $x + 2y - 2z = \alpha$ is 5 $\Rightarrow$ $\alpha = 10$.

Equation of PQ, $\frac{x - 1}{1} = \frac{y + 2}{2} = \frac{z - 1}{-2} = t$

$Q \equiv (t + 1, 2t - 2, -2t + 1)$ and $PQ = 5$

$\Rightarrow$ $t = \frac{5 + \alpha}{9} = \frac{5}{3} \Rightarrow Q \equiv \left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$

4. (1)



Direction ratios of QR are 1, 4, 1

Coordinate of $P \equiv \left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right)$

Direction ratios of PT are 2, 2, –1

Angle between QR and PT is 45°

And $PT = 1$

$\Rightarrow \quad PS = TS = \frac{1}{\sqrt{2}}$

5. (4) Any point B on line is $(2\lambda - 2, -\lambda - 1, 3\lambda)$

Point B lies on the plane for some λ

$\Rightarrow \quad (2\lambda - 2) + (-\lambda - 1) + 3\lambda = 3 \quad$ or $\quad \lambda = 3/2$

$\Rightarrow \quad B \equiv (1, -5/2, 9/2)$

The foot of the perpendicular from point $(-2, -1, 0)$ on the plane is the point $A\ (0, 1, 2)$

$\Rightarrow$ Direction ratio of $AB = \left(1, \frac{-7}{2}, \frac{5}{2}\right) \equiv (2, -7, 5)$.

Hence, feet of perpendicular lies on the line $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

6. (3) Let image (x, y, z) of

$$\therefore \frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-7}{1} = -2\left(\frac{3-1+7-3}{1^2+1^2+1^2}\right) = -4$$

$P\ (x, y, z) = (-1, 5, 3)$

Plane passing through $P(-1, 5, 3)$ is

$a\ (x + 1) + b\ (y - 5) + c\ (z - 3) = 0 \qquad (1)$

Also plane contains line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$

So, (0, 0, 0) satisfies the plane (1).

$\Rightarrow \quad a - 5b - 3c = 0 \qquad (2)$

and $a + 2b + c = 0 \qquad (3)$

From (2) and (3), $a = \frac{1}{7}c, b = -\frac{4}{7}c.$

Put in (1)

$$(x + 1) - 4\ (y - 5) + 7\ (z - 3) = 0$$

or $\quad x - 4y + 7z = 0$

7. (1) Given that required plane is perpendicular to the given two planes.

Therefore, the normal vector of required plane is perpendicular to the normal to the given planes.

Therefore, the normal vector of required plane is parallel to the vector:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix} = -14\hat{i} - 2\hat{j} - 15\hat{k}$$

Thus, the equation of required plane passing through (1, 1, 1) will be:

$$-14(x - 1) - 2(y - 1) - 15(z - 1) = 0$$

$\Rightarrow \quad 14x + 2y + 15z = 31$

Multiple Correct Answers Type

1. (1), (4)

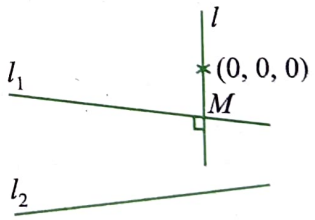
Lines $\frac{x-5}{0} = \frac{y-0}{3-\alpha} = \frac{z-0}{-2}$ and $\frac{x-\alpha}{0} = \frac{y}{-1} = \frac{z}{2-\alpha}$ will be coplanar if shortest distance is zero. Thus,

$$\begin{vmatrix} 5-\alpha & 0 & 0 \\ 0 & 3-\alpha & -2 \\ 0 & -1 & 2-\alpha \end{vmatrix} = 0$$

or $\quad (5 - \alpha)\ (\alpha^2 - 5\alpha + 4) = 0,\ \alpha = 1, 4, 5$

or $\quad \alpha = 1, 4$

2. (2), (4)



The common perpendicular is along

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -2\hat{i} + 3\hat{j} - 2\hat{k}$$

Let $\quad M \equiv (2\lambda, -3\lambda, 2\lambda)$

So, $\quad \frac{2\lambda - 3}{1} = \frac{-3\lambda + 1}{2} = \frac{2\lambda - 4}{2}$

$\Rightarrow \quad \lambda = 1$

So, $\quad M \equiv (2, -3, 2)$

Let the required point be P.

Given that $PM = \sqrt{17}$

$\Rightarrow \quad (3 + 2s - 2)^2 + (3 + 2s + 3)^2 + (2 + s - 2)^2 = 17$

or $9s^2 + 28s + 20 = 0$ or $s = -2, -\frac{10}{9}$.

So, $P \equiv (-1, -1, 0)$ or $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

3. (1), (2)

Let $\vec{v}$ be the vector along L.

Clearly, line L is parallel to given planes

$P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$

$$\therefore \quad \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \hat{i} - 3\hat{j} - 5\hat{k}$$

Since L is passing through origin, any point on line L is $A(\lambda, -3\lambda, -5\lambda)$.

Foot of perpendicular from A to P_1, is

$$\frac{h-\lambda}{1} = \frac{k+3\lambda}{2} = \frac{l+5\lambda}{-1} = -\frac{(\lambda - 6\lambda + 5\lambda + 1)}{1+4+1} = -\frac{1}{6}$$

$$\therefore \quad h = \lambda - \frac{1}{6}, k = -3\lambda - \frac{1}{3}, l = -5\lambda + \frac{1}{6}$$

So, foot or point on locus M is

$$\left(\lambda - \frac{1}{6}, -3\lambda - \frac{1}{3}, -5\lambda + \frac{1}{6}\right).$$

So points (1) and (2) lies on this locus.

4. (2), (4)

Clearly, plane P_3 is $P_2 + \lambda P_1 = 0$.

$\Rightarrow \quad x + \lambda y + z - 1 = 0$

Distance of this plane from point (0, 1, 0) is 1.

$$\Rightarrow \quad \frac{0 + \lambda + 0 - 1}{\sqrt{1 + \lambda^2 + 1}} = \pm 1$$

$$\therefore \quad \lambda = -\frac{1}{2}$$

Thus, equation of P_3 is $2x - y + 2z - 2 = 0$.

Distance of this plane from point (α, β, γ) is 2.

$$\Rightarrow \quad \left|\frac{2\alpha - \beta + 2\gamma - 2}{3}\right| = 2$$

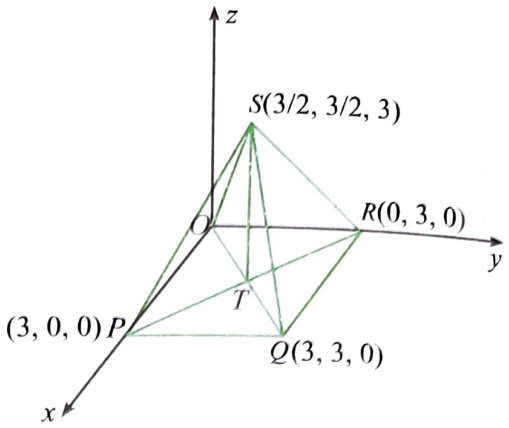
$$\Rightarrow \quad 2\alpha - \beta + 2\gamma = 2 \pm 6$$

Thus, options (2), (4) are correct.

5. (2), (3), (4)

Direction ratios of $OQ \equiv (3, 3, 0)$

Direction ratios of $OS \equiv \left(\frac{3}{2}, \frac{3}{2}, 3\right)$



If θ is the angle between OS and OQ, then

$$\cos\theta = \frac{3 \times \frac{3}{2} + 3 \times \frac{3}{2}}{\sqrt{3^2 + 3^2}\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 3^2}} = \frac{1}{\sqrt{3}}$$

Hence (1) is wrong.

Normal to plane OQS,

$$\overrightarrow{OQ} \times \overrightarrow{OS} = \pm \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 3 & 0 \\ \frac{3}{2} & \frac{3}{2} & 3 \end{vmatrix} = \pm(9\hat{i} - 9\hat{j})$$

$\therefore$ Equation of plane OQS is

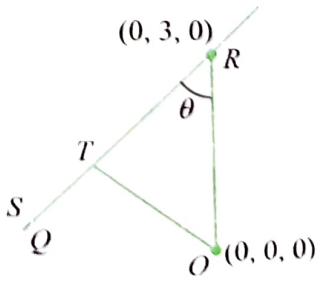
$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (9\hat{i} - 9\hat{j}) = 0$$

$\Rightarrow \quad x - y = 0$

Distance of P from plane OQS = distance PT = distance $OT = \frac{3}{\sqrt{2}}$

Equation of RS is $\frac{x-0}{\frac{3}{2}-0} = \frac{y-3}{\frac{3}{2}-3} = \frac{z-0}{3-0}$

or $\frac{x}{\frac{3}{2}} = \frac{y-3}{-\frac{3}{2}} = \frac{z}{3}$



Angle between line RS and OR

$$\cos\theta = \frac{0+3\left(-\frac{3}{2}\right)+0}{3\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2+3^2}} = -\frac{1}{\sqrt{6}}$$

$$\text{Distance} = OT = OR\sin\theta = 3\sqrt{1-\frac{1}{6}} = 3\sqrt{\frac{5}{6}} = \sqrt{\frac{15}{2}}$$

6. (3), (4)

(1) Given planes $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$

Normal to plane P_1, $\vec{n_1} = 2\hat{i} + \hat{j} - \hat{k}$

Normal to plane P_2, $\vec{n_2} = \hat{i} + 2\hat{j} + \hat{k}$

Vector along the line of intersection is $\vec{n_1} \times \vec{n_2}$.

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(\hat{i} - \hat{j} + \hat{k})$$

(2) $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$

or $\frac{x-4/3}{3} = \frac{y-1/3}{-3} = \frac{z}{3}$

This line is along the vector $3(\hat{i} - \hat{j} + \hat{k})$.

Thus, the line is parallel to the line of intersection of given planes P_1 and P_2.

(3) Angle between planes P_1 and P_2 is given by

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{2+2-1}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$\Rightarrow \theta = 60°$

(4) Plane P_3 is perpendicular to the line of intersection of planes P_1 and P_2.

Also, P_3 passes through the point (4, 2, –2).

So, equation of P_3 is

$$1(x-4) - (y-2) + 1(z+2) = 0$$

or $P_3 : x - y + z = 0$

Distance of (2, 1, 1) from $P_3 = \left|\frac{2-1+1}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$

Matrix Match Type

1. a → r; b → q; c → s; d → p

Plane perpendicular to P_1 and P_2 has direction ratios of normal

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 1 & 2 \\ 3 & 5 & -6 \end{vmatrix} = -16\hat{i} + 48\hat{j} + 32\hat{k} \quad \text{(i)}$$

For point of intersection of lines

$$(2\lambda_1 + 1, -\lambda_1, \lambda_1 - 3) \equiv (\lambda_2 + 4, \lambda_2 - 3, 2\lambda_2 - 3)$$

$\Rightarrow$ $2\lambda_1 + 1 = \lambda_2 + 4$ or $2\lambda_1 - \lambda_2 = 3$

and $-\lambda_1 = \lambda_2 - 3$ or $\lambda_1 + \lambda_2 = 3$

$\Rightarrow$ $\lambda_1 = 2, \lambda_2 = 1$

$\therefore$ Point is (5, –2, –1) (ii)

From (i) and (ii), required planes is

$$-1(x-5) + 3(y+2) + 2(z+1) = 0$$

or $x - 3y - 2z = 13$

$\Rightarrow$ $a = 1, b = -3, c = -2, d = 13$

Numerical Value Type

1. (6) Let normal to plane be $l\hat{i} + m\hat{j} + n\hat{k}$

$$2l + 3m + 4n = 0$$

and $3l + 4m + 5n = 0$

$$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

Equation of plane will be

$$a(x-1) + b(y-2) + c(z-3) = 0$$

or $-1(x-1) + 2(y-2) - 1(z-3) = 0$

or $-x + 1 + 2y - 4 - z + 3 = 0$

or $-x + 2y + z = 0$

or $x - 2y + z = 0$

or $\frac{|d|}{\sqrt{6}} = \sqrt{6}$

or $d = 6$

2. (8)

Let $P \equiv (\alpha, \beta, \gamma)$.

R is image of P in xy-plane. So, $R \equiv (\alpha, \beta, -\gamma)$.

$Q(x, y, z)$ is the image of P in plane $x + y - 3 = 0$.

$$\therefore \quad \frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{0} = \frac{-2(\alpha+\beta-3)}{2}$$

$\therefore$ $x = 3 - \beta, y = 3 - \alpha, z = \gamma$

Point $Q(3 - \beta, 3 - \alpha, \gamma)$ lies on z-axis.

$\therefore$ $\beta = 3, \alpha = 3$

$\therefore$ $P \equiv (3, 3, \gamma)$ whose distance from x-axis is 5.

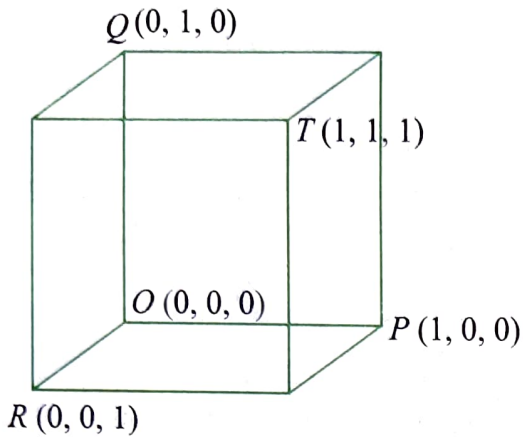
$\therefore$ $9 + \gamma^2 = 25$

So, $\gamma^2 = 16 \Rightarrow \gamma = 4$

Hence $P \equiv (3, 3, 4)$ and $R \equiv (3, 3, -4)$.

$\therefore$ $PR = 8$

3. (0.5)



Point $S \equiv (1/2, 1/2, 1/2)$

Point $T \equiv (1, 1, 1)$

$$\vec{p} = \overrightarrow{SP} = \frac{\hat{i} - \hat{j} - \hat{k}}{2}$$

$$\vec{q} = \overrightarrow{SQ} = \frac{-\hat{i} + \hat{j} - \hat{k}}{2}$$

$$\vec{r} = \overrightarrow{SR} = \frac{-\hat{i} - \hat{j} + \hat{k}}{2}$$

$$\vec{t} = \overrightarrow{ST} = \frac{\hat{i} + \hat{j} + \hat{k}}{2}$$

Now

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{vmatrix} \times \frac{1}{4} = \frac{1}{4}(2\hat{i} + 2\hat{j}) = \frac{\hat{i} + \hat{j}}{2}$$

$$\vec{r} \times \vec{t} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \times \frac{1}{4} = \frac{-2\hat{i} + 2\hat{j}}{4} = \frac{-\hat{i} + \hat{j}}{2}$$

$$\text{Now, } (\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} \times \frac{1}{4} = \frac{\hat{k}}{2}$$

$$\Rightarrow |(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})| = \frac{1}{2} = 0.5$$

1 APPENDIX

Chapterwise Solved January 2019 JEE Main Questions (All Sets)

Chapter 2 (Introduction to Vectors)

1. Let $\vec{\alpha} = (\lambda - 2)\vec{a} + \vec{b}$ and $\vec{\beta} = (4\lambda - 2)\vec{a} + 3\vec{b}$ be two given vectors where vectors $\vec{a}$ and $\vec{b}$ are non-collinear. The value of λ for which vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear is

(1) -3 (2) 4

(3) 3 (4) -4

2. Let $\sqrt{3}\hat{i} + \hat{j}$, $\hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1 - \beta)\hat{j}$, respectively, be the position vectors of the points A, B and C with respect to the origin O. If the distance of C from the bisector of the acute angle between OA and OB is $\frac{3}{\sqrt{2}}$, then the sum of all possible values of β is

(1) 2 (2) 1

(3) 3 (4) 4

3. The sum of the distinct real values of μ, for which the vectors $\mu\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + \mu\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + \mu\hat{k}$ are coplanar is

(1) 2 (2) 0

(3) -1 (4) 1

4. Let S be the set of all real values of λ such that a plane passing through the points $(-\lambda^2, 1, 1)$, $(1, -\lambda^2, 1)$ and $(1, 1, -\lambda^2)$ also passes through the point $(-1, -1, 1)$. Then S is equal to

(1) $\{\sqrt{3}\}$ (2) $\{\sqrt{3}, -\sqrt{3}\}$

(3) $\{1, -1\}$ (4) $\{3, -3\}$

Chapter 3 (Different Products of Vectors)

5. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c}$ be vectors such that $\vec{a} \times \vec{c} + \vec{b} = \vec{0}$ and $\vec{a} \cdot \vec{c} = 4$. Then $|\vec{c}|^2$ is equal to

(1) $\frac{19}{2}$ (2) 8

(3) $\frac{17}{2}$ (4) 9

6. Let $\vec{a} = \hat{i} + \hat{j} + \sqrt{2}\hat{k}$, $b = b_1\hat{i} + b_2\hat{j} + \sqrt{2}\hat{k}$ and $\vec{c} = 5\hat{i} + \hat{j} + \sqrt{2}\hat{k}$ be three vectors such that the projection vector of $\vec{b}$ on $\vec{a}$ is $\vec{a}$. If $\vec{a} + \vec{b}$ is perpendicular to $\vec{c}$, then $|\vec{b}|$ is equal to

(1) $\sqrt{22}$ (2) 4

(3) $\sqrt{32}$ (4) 6

7. Let $\vec{a} = 2\hat{i} + \lambda_1\hat{j} + 3\hat{k}$, $\vec{b} = 4\hat{i} + (3 - \lambda_2)\hat{j} + 6\hat{k}$ and $\vec{c} = 3\hat{i} + 6\hat{j} + (\lambda_3 - 1)\hat{k}$ be three vectors such that $\vec{b} = 2\vec{a}$ and $\vec{a}$ is perpendicular to $\vec{c}$. Then a possible value of $(\lambda_1, \lambda_2, \lambda_3)$ is

(1) (1/2, 4, –2) (2) (–1/2, 4, 0)
(3) (1, 3, 1) (4) (1, 5, 1)

8. Let $\vec{a}=\hat{i}+2\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+\lambda\hat{j}+4\hat{k}$ and $\vec{c}=2\hat{i}+4\hat{j}+\left(\lambda^2-1\right)\hat{k}$ be coplanar vectors. Then the non-zero vector $\vec{a}\times\vec{c}$ is

(1) $-14\hat{i}-5\hat{j}$ (2) $-10\hat{i}-5\hat{j}$
(3) $-10\hat{i}+5\hat{j}$ (4) $-14\hat{i}+5\hat{j}$

9. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three unit vectors, out of which vectors $\vec{b}$ and $\vec{c}$ are non-parallel. If α and β are the angles which vector $\vec{a}$ makes with vectors $\vec{b}$ and $\vec{c}$, respectively, and $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\frac{1}{2}\vec{b}$, then $|\alpha-\beta|$ is equal to

(1) 60° (2) 30°
(3) 90° (4) 45°

Chapter 4 (Line and Plane)

10. The plane through the intersection of the planes $x+y+z=1$ and $2x+3y-z+4=0$ and parallel to y-axis also passes through the point

(1) (–3, 0, –1) (2) (3, 3, –1)
(3) (3, 2, 1) (4) (–3, 1, 1)

11. The equation of the line passing through (–4, 3, 1), parallel to the plane $x+2y-z-5=0$ and intersecting the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z-2}{-1}$ is

(1) $\frac{x+4}{-1}=\frac{y-3}{1}=\frac{z-1}{1}$

(2) $\frac{x+4}{3}=\frac{y-3}{-1}=\frac{z-1}{1}$

(3) $\frac{x+4}{1}=\frac{y-3}{1}=\frac{z-1}{3}$

(4) $\frac{x-4}{2}=\frac{y+3}{1}=\frac{z+1}{4}$

12. The equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is

(1) $x+2y-2z=0$
(2) $x-2y+z=0$
(3) $5x+2y-4z=0$
(4) $3x+2y-3z=0$

13. If the lines $x=ay+b, z=cy+d$ and $x=a'z+b'$, $y=c'z+d'$ are perpendicular, then

(1) $cc'+a+a'=0$
(2) $aa'+c+c'=0$
(3) $ab'+bc'+1=0$
(4) $bb'+cc'+1=0$

14. The plane passing through the point (4, –1, 2) and parallel to the lines $\frac{x+2}{3}=\frac{y-2}{-1}=\frac{z+1}{2}$ and $\frac{x-2}{1}=\frac{y-3}{2}=\frac{z-4}{3}$ also passes through the point

(1) (–1, –1, –1) (2) (–1, –1, 1)
(3) (1, 1, –1) (4) (1, 1, 1)

15. Let A be a point on the line $\vec{r}=(1-3\mu)\hat{i}+(\mu-1)\hat{j}+(2+5\mu)\hat{k}$ and $B(3,2,6)$ be a point in the space. Then the value of μ for which the vector $\overrightarrow{AB}$ is parallel to the plane $x-4y+3z=1$ is

(1) $\frac{1}{2}$ (2) $-\frac{1}{4}$
(3) $\frac{1}{4}$ (4) $\frac{1}{8}$

16. The plane which bisects the line segment joining the points (–3, –3, 4) and (3, 7, 6) at right angles,

passes through which one of the following points?

(1) $(4, -1, 7)$ (2) $(4, 1, -2)$

(3) $(-2, 3, 5)$ (4) $(2, 1, 3)$

17. On which of the following lines lies the point of intersection of the line, $\frac{x-4}{2} = \frac{y-5}{2} = \frac{z-3}{1}$ and the plane, $x + y + z = 2$?

(1) $\frac{x-2}{2} = \frac{y-3}{2} = \frac{z+3}{3}$

(2) $\frac{x-4}{1} = \frac{y-5}{1} = \frac{z-5}{-1}$

(3) $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z+4}{-5}$

(4) $\frac{x+3}{3} = \frac{4-y}{3} = \frac{z+1}{-2}$

18. The direction ratios of normal to the plane through the points $(0, -1, 0)$ and $(0, 0, 1)$ and making an angle $\frac{\pi}{4}$ with the plane $y - z + 5 = 0$ are

(1) $2\sqrt{3}, 1, -1$ (2) $2, \sqrt{2}, -\sqrt{2}$

(3) $2, -1, 1$ (4) $\sqrt{2}, 1, 1$

19. The plane containing the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z-1}{3}$ and also containing its projection on the plane $2x + 3y + z = 5$, contains which of the following points?

(1) $(2, 0, -2)$ (2) $(-2, 2, 2)$

(3) $(0, -2, 2)$ (4) $(2, 2, 0)$

20. If the point $(2, \alpha, \beta)$ lies on the plane which passes through the points $(3, 4, 2)$ and $(7, 0, 6)$ and is perpendicular to the plane $2x - 5y = 15$, then $2\alpha - 3\beta$ is equal to

(1) 5 (2) 17

(3) 12 (4) 7

21. Two lines $\frac{x-3}{1} = \frac{y+1}{3} = \frac{z-6}{-1}$ and $\frac{x+5}{7} = \frac{y-2}{-6} = \frac{z-3}{4}$ intersect at the point R. The reflection of R in the xy-plane has coordinates

(1) $(2, 4, 7)$ (2) $(-2, 4, 7)$

(3) $(2, -4, -7)$ (4) $(2, -4, 7)$

22. The perpendicular distance from the origin to the plane containing the two lines $\frac{x+2}{3} = \frac{y-2}{5} = \frac{z+5}{7}$ and $\frac{x-1}{1} = \frac{y-4}{4} = \frac{z+4}{7}$ is

(1) $11/\sqrt{6}$ (2) $6\sqrt{11}$

(3) 11 (4) $11\sqrt{6}$

23. A tetrahedron has vertices $P(1, 2, 1)$, $Q(2, 1, 3)$, $R(-1, 1, 2)$ and $O(0, 0, 0)$. The angel between the faces OPQ and PQR is

(1) $\cos^{-1}\left(\frac{9}{35}\right)$ (2) $\cos^{-1}\left(\frac{19}{35}\right)$

(3) $\cos^{-1}\left(\frac{17}{31}\right)$ (4) $\cos^{-1}\left(\frac{7}{31}\right)$

24. If angle between the line $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$ and the plane $x - 2y - kz = 3$ is $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$, then a value of k is

(1) $-\frac{5}{3}$ (2) $\sqrt{\frac{3}{5}}$

(3) $\sqrt{\frac{5}{3}}$ (4) $-\frac{3}{5}$

Answers Key

1. (4)	2. (2)	3. (3)	4. (2)	13. (2)	14. (4)	15. (3)	16. (2)
5. (1)	6. (4)	7. (2)	8. (3)	17. (3)	18. (2)	19. (1)	20. (4)
9. (2)	10. (3)	11. (2)	12. (2)	21. (3)	22. (1)	23. (1)	24. (3)